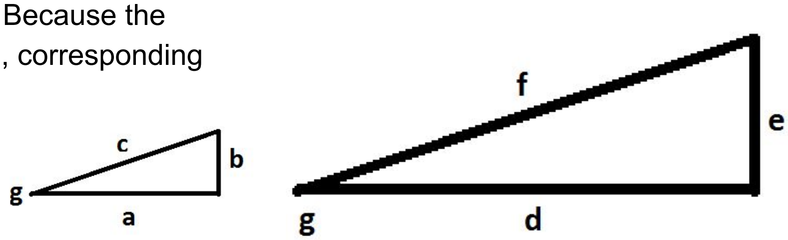
Trigonometry

Introduction

Trigonometry is the study of the relationship between the sides and the angles of triangles. Using trig one can, say, estimate a building's height from the length of its shadow and the angle of the Sun, or estimate the distance to a point across a river by the length of a distance on one shore and the angle to the point on the opposite shore.

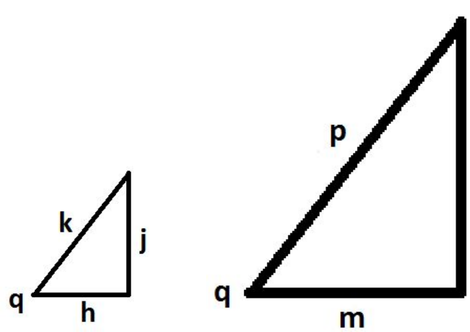
But it only starts there. Trigonometry also deals with how distances change with revolutions. It can describe, for example, the way the length of a day changes with latitude and with the season or the way the height above the ground of a spot on a tire changes with the tire's rotation.

Let's look into how the sides and angles of right triangles relate. Consider two right triangles of the same shape but different sizes. Because the triangles are of are the same shape, corresponding angles will have the same measure. And the lengths of corresponding sides will have the same ratios. Because the two triangles to the right are of the



same shape, the ratio of the length of side a to the length of side c is the same as the ratio of the length of side d to the length of side f.

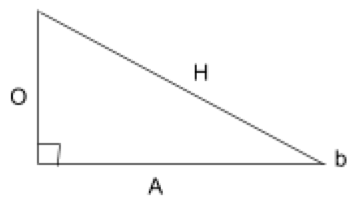
Now consider another two right triangles, different in shape from the first two but the same in shape as each other, and still of different sizes



. As above, the lengths of corresponding sides will have the same ratios but those ratios will be different from the ratios of the first two triangles. For example, the ratio of the length of side h to the length of side k is the same as the ratio of the length of side m to the length of side p, but that is different from ratio of the length of side a to the length of side c in the first pair of triangles.

Sine, Cosine, and Tangent

Right Triangle Definitions



Sine, cosine, and tangent are ratios that relate the lengths of the sides of right triangles. They are functions of an angle. In other words, we talk about the sine, cosine, and tangent of an angle.

The sine of angle b in the diagram is equal to the ratio of the opposite side to the hypotenuse: $sin\ b\ =\ O/H$.

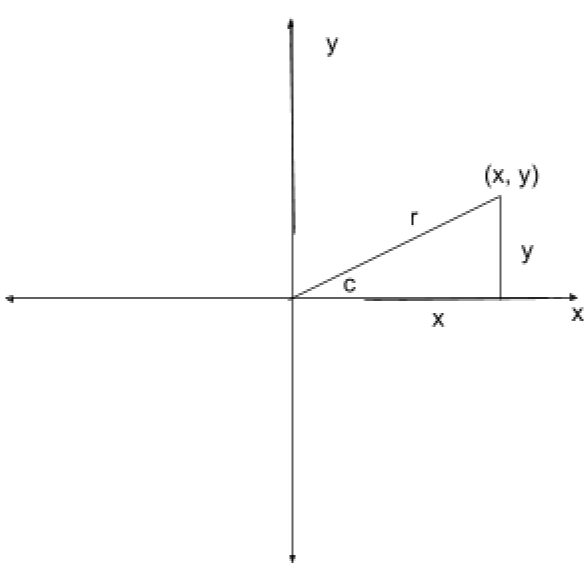
The cosine of angle b is equal to the ratio of the adjacent side to the hypotenuse: $cos\ b\ =\ A/H$.

The tangent of angle b is equal to the ratio of the opposite side to the adjacent side: $tan\ b\ =\ O/A$.

These definitions work for any angle b that is greater than 0 and less than 90 degrees in a right triangle.

One way to remember these definitions is SOH-CAH-TOA: **S**in**=O**pposite**/H**ypotenuse**, C**os**=A**djacent**/H**ypotenuse, **T**an**=O**pposite**/A**djacent. If you can't remember SOH-CAH-TOA and if you like horses (or even if you don't), try this: **S**ome **O**ld **H**orse **- C**aught **A**nother **H**orse **- T**aking **O**ats **A**gain.

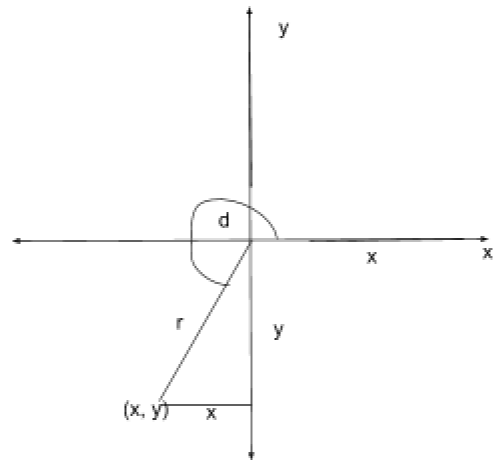
Coordinate Plane Definitions



The right triangle definitions above work only for angles that measure more than 0 and less than 90 degrees. But trig functions can be useful for larger angles as well. To use trig functions for larger angles requires new definitions for those functions.

Consider an angle in the first quadrant (upper right quadrant) of the coordinate plane.

Given a right triangle with one angle (not the right angle; call it angle c) at the origin, and given that the other angle that is not the right angle is located at a point with coordinates x and y a distance r from the origin, then, looking from angle c, y is the opposite side, x is the adjacent side, r is the hypotenuse, and by the right triangle definitions above, $\sin c = y/r$, $\cos c = x/r$, and $\tan c = y/x$. These definitions of sine, cosine, and tangent can be extended all around the coordinate plane. For example, in the diagram to the right, x/r is the cosine, y/r is the sine, and y/x is the tangent of angle d.

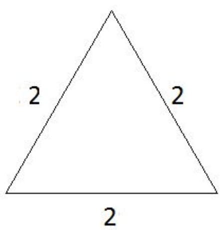


Special Angles

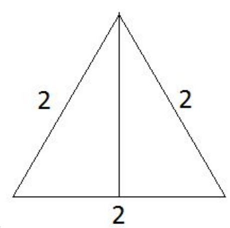
There is no easy way to find the sine, cosine, or tangent of most angles, but there are some we can get our arms around: 30 degrees, 45 degrees, and 60 degrees.

Thirty Degrees and Sixty Degrees

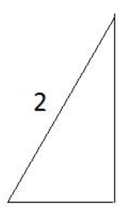
Consider an equilateral triangle with sides of length 2:



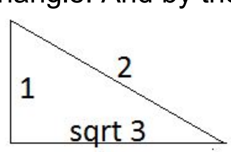
Draw an altitude from the apex to the opposite base and break this equilateral triangle into two congruent right triangles



. Look at one of them.

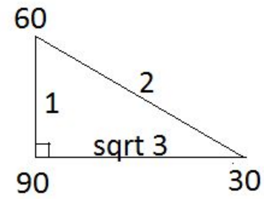


Its hypotenuse should have length 2, because that hypotenuse was a side of the original triangle. The new triangle's shorter leg should have length 1, because it's half of what was a side of the original triangle. And by the Pythagorean Theorem, its longer leg should



have length $\sqrt{2^2 - 1^2} = \sqrt{3}$.

Now consider this triangle's angles. The two legs meet at a 90-degree angle at the point where the altitude met the base of the original triangle. The greater of the two remaining angles is an angle of the original triangle, each of whose angles measures 60 degrees, so its measure must be 60 degrees. And the lesser of those two angles is half of what was an angle of the original triangle, so its measure must be 30 degrees.

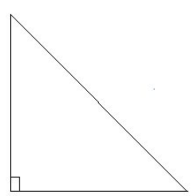


Now we've got what we need to find the sine, cosine, and tangent of 30 degrees and 60 degrees. Using right-triangle trig definitions, you can see that:

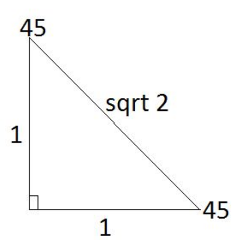
- $sin\ 30 = opposite/hypotenuse = 1/2$
- $cos\ 30 = adjacent/hypotenuse = \sqrt{3}/2$
- $tan\ 30 = opposite/adjacent = 1/\sqrt{3} = \sqrt{3}/3$
- $sin\ 60 = opposite/hypotenuse = \sqrt{3}/2$
- $cos\ 60 = adjacent/hypotenuse = 1/2$
- $tan\ 60 = opposite/adjacent = \sqrt{3}/1 = \sqrt{3}$

Forty-Five Degrees

Let's look the sine, cosine, and tangent functions for one more angle: 45 degrees. To do that, consider an isosceles right triangle.



If the length of each leg is 1, then by the Pythagorean Theorem the length of the hypotenuse is $\sqrt{1^2 + 1^2} = \sqrt{2}$.



From the sketch, looking from either of the 45-degree angles, you can see that:

- $sin\ 45 = opposite/hypotenuse = 1/\sqrt{2} = \sqrt{2}/2$
- $cos\ 45 = adjacent/hypotenuse = 1/\sqrt{2} = \sqrt{2}/2$
- $tan\ 45 = opposite/adjacent = 1/1 = 1$

Radians

The *radian* is a unit of angle measure. It is used often in trig.

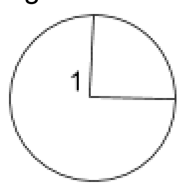
There are 2π radians in a circle. There are also 360 degrees in a circle. That means 360 degrees = 2π radians, or $360/2\pi = 1;$ therefore $180/\pi = \pi/180 = 1.$ That is, you can use $180/\pi$ or

$\pi/180$ as a conversion factor to convert between radians and degrees, or between degrees and radians.

Here are degree-radian conversions of some commonly used angles.

Degrees	Radians
0	0
30	$\pi/6$
45	$\pi/4$
60	$\pi/3$
90	$\pi/2$
180	π
270	$3\pi/2$
360	2π

Note that for a circle with a radius of 1, the arc length between two points on the circle is equal to the measure in radians of the central angle formed by line segments extending from the points on the circle to the circle's center.

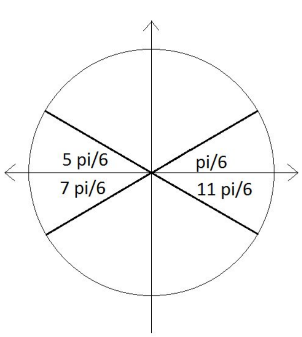


For example, in this circle with a radius of 1, if the angle formed by the two radii shown is 90 degrees, which is $\pi/2$ radians, then the arc between those points where the radii touch the circle is $1/4$ of the circle. The length of the arc between those points is $1/4$ the circumference of the circle; that is, the arc length if $2\pi r/4 = 2\pi(1)/4 = \pi/2$ -- the same as the angle measure, in radians, at that point.

Reference Angles

The right triangle special angles tell us about the sine, cosine, and tangent functions in the first quadrant. What happens in the other quadrants?

Let's look an angle 30 degrees, or $\pi/6$ radians, with respect to the x-axis. In the other quadrants are angles that make the same angle with the x-axis: $5\pi/6$ in the second quadrant, $7\pi/6$ in the third quadrant, and $11\pi/6$ in the fourth quadrant. The first-quadrant angle $\pi/6$ is called a *reference angle* to these other angles.



What happens to the trig functions in these places? On the circle at all these angles -- $\pi/6, 5\pi/6, 7\pi/6,$ and $11\pi/6$ -- the absolute value of y divided by the circle's radius, $|y|/r,$ is the same. Since y/r is the sine of the angle and since r is always positive, $|y|/r = |sin\ \theta|$ (where θ is the angle's name). So now we know at least the absolute value of the sines of three more angles. What about their signs?

Again, $sin\theta = y/r$ and r is always positive; therefore, in each quadrant $sin\theta$ takes the same sign as y. Recall that $sin\ \pi/6\ =\ 1/2$. So $sin\ 5\pi/6\ =\ 1/2$ (positive because y is positive in the second quadrant), $sin 7\pi/6\ =\ -1/2$ (negative because y is negative in the third quadrant), and $sin\ 11\pi/6\ =\ -1/2$ (negative because y is negative in the fourth quadrant).

We can look at values of cosine at the same points. Recall that $cos\theta = x/r$ and r is always positive; therefore, in each quadrant $cos\theta$ takes the same sign as x. Recall that $cos\ \pi/6\ =\ \sqrt{3}/2$. So $cos\ 5\pi/6\ =\ -\sqrt{3}/2$ (negative because x is negative in the second quadrant), $cos\ 7\pi/6\ =\ -\sqrt{3}/2$ (negative because x is negative in the third quadrant), and $cos\ 11\pi/6\ =\ \sqrt{3}/2$ (positive because x is positive in the fourth quadrant).

And tangent? Recall that $tan\theta = y/x$; therefore, $tan\theta$ is positive where x and y have the same sign (quadrant I, where both are positive, and quadrant III, where both are negative) and $tan\theta$ is negative where x and y have opposite signs (quadrant II, where x is negative and y is positive, and quadrant IV, where x is positive and y is negative). Recall that $tan\ 30 = tan\ \pi/6 = \sqrt{3}/3$. So $tan\ 5\pi/6\ =\ -\sqrt{3}/3$, $tan 7\ \pi/6\ =\ \sqrt{3}/3$, and $tan\ 11\pi/6\ =\ -\sqrt{3}/3$.

We can do the same for the other reference angles for which we have exact values of sine, cosine, and tangent; that is, for 45 degrees ($\pi/4$) and 60 degrees ($\pi/3$). But first let's get a sense of how the three functions change as we go around the circle.

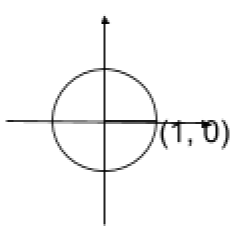
Sine, Cosine, and Tangent around the Circle

As stated above, the coordinate plane definitions of the sine, cosine, and tangent of angle θ are

$sin\ \theta = y/r$

$cos\ \theta = x/r$ and

$tan\ \theta = y/x$.



At the point on the circle where $\theta = 0$, $y = 0$ (because there is no distance above the x-axis) and $x = r$ (because the x-value at this point is the same as r, the length of the line segment from the origin to the circle). So $sin\ 0 = y/r = 0$, $cos\ 0 = x/r = 1$, and $tan\ 0 = y/x = 0$.

As angle θ goes from 0 to 360 degrees, the *initial side* of angle θ, that is the side that begins at the origin and extends to the right along the x-axis, remains where it is, and the *terminal side*, that is side r, sweeps around the circle going counterclockwise.

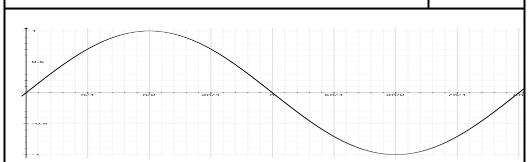
If $r = 1$, then $sin\ \theta = y$ and $cos\ \theta = x$. That means that when r=1, the x and y coordinates at the end of side r are (cos θ, sin θ).

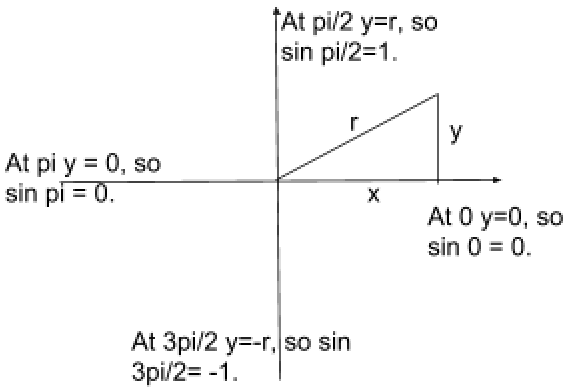
Sine

What happens to the sine function as we go around the circle? As we rotate counterclockwise through the first quadrant, the ratio y/r, which is sine, increases until we get to $\pi/2$ radians. At $\pi/2$ the terminal side, r, coincides with the y-axis so that $y/r = 1$, so we can say $sin\ \pi/2 = 1.$ The sine function is at its maximum here, because this is where y is greatest. As we continue into the second quadrant from $\pi/2$ to π, the value of sine decreases until it is once again 0 at $y = \pi.$ Until now, sine has been positive, because we've been working in the first and second quadrants, where y is positive. Now as we cross the x-axis, we are moving into territory where y, and therefore sine, is negative. Here y continues to become more negative until we get to $3\pi/2$ radians, where once again r coincides with the y-axis -- but this time y is negative, so $sin\ 3\pi/2 \ =-1.$ (We always take r as positive.) Then, in the fourth quadrant, y remains negative and its absolute value decreases until we get to 2π radians (360 degrees), the end of our journey, where sine is once again 0 because, after all, 2π is at the same place on the circle as is 0, where sine is 0.

0, $\pi/2$, π, and $3\pi/2$ are called *quadrantal angles.*

θ	$sin\ \theta$
0 degrees, 0 radians	0
$\pi/2$ radians, 90 degrees	1
π radians, 180 degrees	0
$3\pi/2$ radians, 270 degrees	-1
2π radians, 360 degrees	0

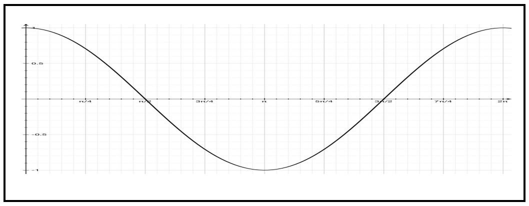


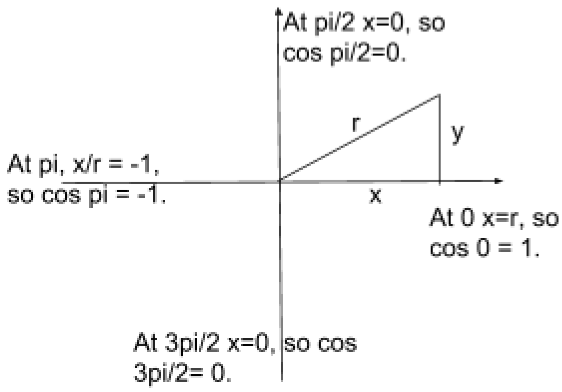


Cosine

Now let's see what happens if we do the same investigation with cosine: We start at $\theta = 0$, where $x = r$, so $cos\ 0 = x/r = 1$. As we rotate counterclockwise through the first quadrant, the ratio x/r, which is cosine, decreases until we get to $\pi/2$ radians. At $\pi/2$ the terminal side, r, coincides with the y-axis so that there is no distance from the y-axis and $x/r = 0$, so we can say $cos\ \pi/2 = 0$. As we cross the y-axis into the second quadrant, cosine becomes negative (because x becomes negative) and cosine continues to decrease from $\pi/2$ to π. At π radians, r coincides with the negative x-axis, and cosine, which is x/r, is -1. As we come back toward the negative y-axis, the absolute value of cosine decreases until at $3\pi/2$ radians, when we reach the y-axis, cosine is 0. Once we cross the y-axis, x, and therefore cosine, becomes positive. Cosine increases until we get to 2π radians (360 degrees), the end of our journey, where cosine is once again 1, its maximum value, because, after all, 2π is at the same place on the circle as is 0, where cosine is 1.

θ	$cos\ \theta$
0 degrees or radians	1
$\pi/2$ radians or 90 degrees	0
π radians or 180 degrees	-1
$3\pi/2$ radians or 270 degrees	0
2π radians or 360 degrees	1





That's the general flow of the sine and cosine curves around the coordinate plane.

Tangent

Now let's look at the tangent function around the circle.

We start at $\theta = 0$, where $y = r$, so $tan\ 0 = y/x = 0$. As we rotate counterclockwise through the first quadrant, the ratio y/x increases until we get to $\pi/2$ radians. At $\pi/2$ the terminal side, r, coincides with the y-axis so that there is no distance from the y-axis and $x = 0$, so $y/x = 1/0$ and $tan\ \theta = y/x$ is undefined. The tangent function has a vertical asymptote here. After we cross the y-axis -- and the asymptote -- into the second quadrant, tangent is negative (because x becomes negative while y remains positive, making the ratio y/x negative). Close to $\pi/2$ tangent is large in absolute value, so it's *very* negative. It decreases in absolute value through the second quadrant. At π radians, r coincides with the negative x-axis and tangent, which is y/x, is 0. Once we cross the x-axis into the third quadrant, x and y are both negative, so tangent is positive. As we come back toward the negative y-axis, x approaches 0 again until at $3\pi/2$ radians, when we reach the y-axis, tangent is once again undefined. Once we cross the y-axis x becomes positive while y is still negative, so tangent is negative once again. From there, tangent decreases in absolute value until we get to 2π radians (360 degrees), the end of our journey, where tangent is once again 0, because, after all, 2π is at the same place on the circle as is 0, where tangent is 0.

θ	$tan\ \theta$
0 degrees or radians	0

$\pi/4$ radians or 45 degrees	1
$\pi/2$ radians or 90 degrees	undefined
$3\pi/4$ radians or 135 degrees	-1
π radians or 180 degrees	0
$5\pi/4$ radians or 225 degrees	1
$3\pi/2$ radians or 270 degrees	undefined
$7\pi/4$ radians or 315 degrees	-1
2π radians or 360 degrees	0

Notice a couple of differences between sine and cosine on the one hand and tangent on the other. Sine and cosine are pretty similar: They both go up and down, with y-values varying between -1 and 1, and both complete their cycles in the space of one revolution around the circle, 2π radians. Tangent, on the other hand, can take on any y-value. It always goes up, except at the vertical asymptotes, where its sign changes abruptly from positive to negative. Tangent goes through its cycle in the space of one-half revolution, or π radians, and then repeats itself.

Putting It All Together

We have found the values of all three functions -- sine, cosine, and tangent -- at the quadrantal angles; we have examined what signs the three functions take in each quadrant, in between quadrantal angles; and we have established that in the quadrants other than the first, the sine, cosine, and tangent of each angle have the same absolute value as does the reference angle in the first quadrant.

With that knowledge we can chart the values of sine, cosine, and tangent at various places around the circle:

Angle θ		$\sin\theta$	$\cos\theta$	$\tan\theta$
Degrees	Radians			
0	0	0	1	0
30	$\pi/6$	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$

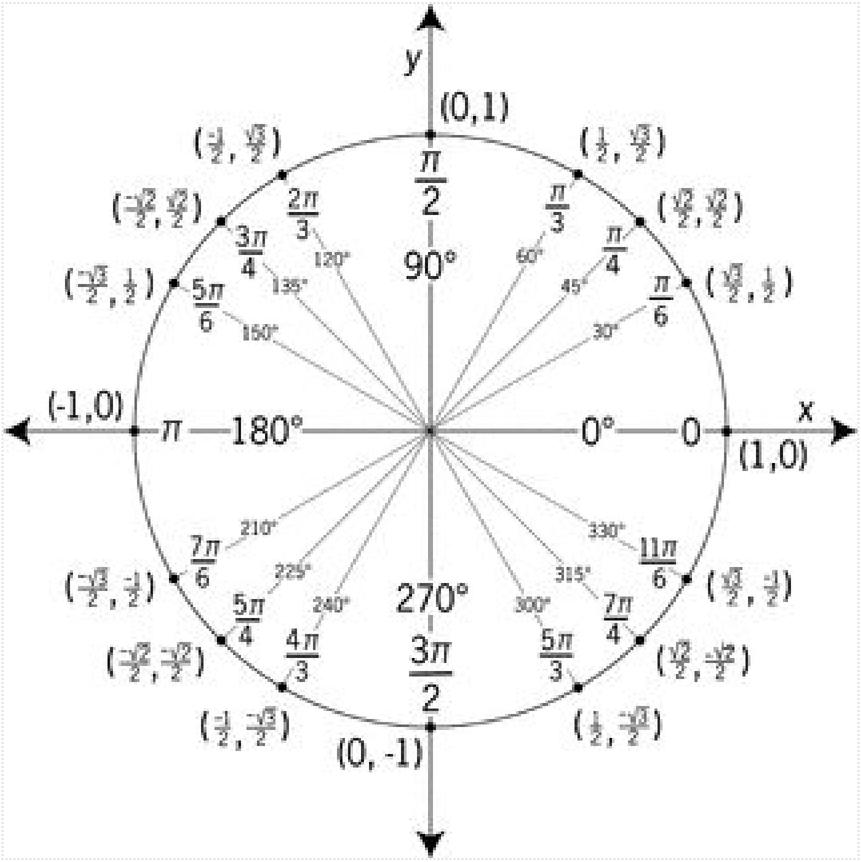
45	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60	$\pi/3$	$\sqrt{3}/2$	1/2	$\sqrt{3}$
90	$\pi/2$	1	0	undefined
120	$2\pi/3$	$\sqrt{3}/2$	-1/2	$-\sqrt{3}$
135	$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1
150	$5\pi/6$	1/2	$-\sqrt{3}/2$	$-\sqrt{3}/3$
180	π	0	-1	0
210	$7\pi/6$	-1/2	$-\sqrt{3}/2$	$\sqrt{3}/3$
225	$5\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1
240	$4\pi/3$	$-\sqrt{3}/2$	-1/2	$\sqrt{3}$
270	$3\pi/2$	-1	0	undefined
300	$5\pi/3$	$-\sqrt{3}/2$	1/2	$-\sqrt{3}$
315	$7\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1
330	$11\pi/6$	-1/2	$\sqrt{3}/2$	$-\sqrt{3}/3$
360	2π	0	1	0

The Unit Circle

The *unit circle* is a useful mechanism for displaying (and helping you remember) the values of the trig functions around the circle. It is a circle with a radius, r, of 1. Recall that:

- $\sin\theta = y/r$;
- $\cos\theta = x/r$; and
- $\tan\theta = y/x$.

Since $r = 1$ on the circle, at any point on the circle $x = \cos\theta$ and $y = \sin\theta$.



The unit circle.

Note that the unit circle gives values of only sine and cosine, not tangent. But since $\sin\theta = y/r$, $\cos\theta = x/r$, and $\tan\theta = y/x$, it’s also true that $\tan\theta = \sin\theta / \cos\theta$, and you can calculate the value of tangent at any point by dividing sine by cosine.

The unit circle can help you visualize the trends of sine, cosine, and tangent around the circle. Another format you will use often is the graph of x v. sinx (or x v. cos x or x v. tan x).

These sketches show the values of $\sin x$ at some values of x that we have examined.

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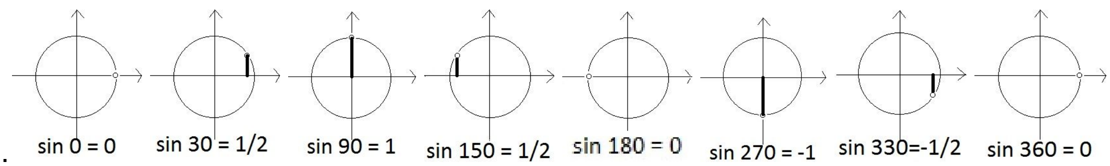
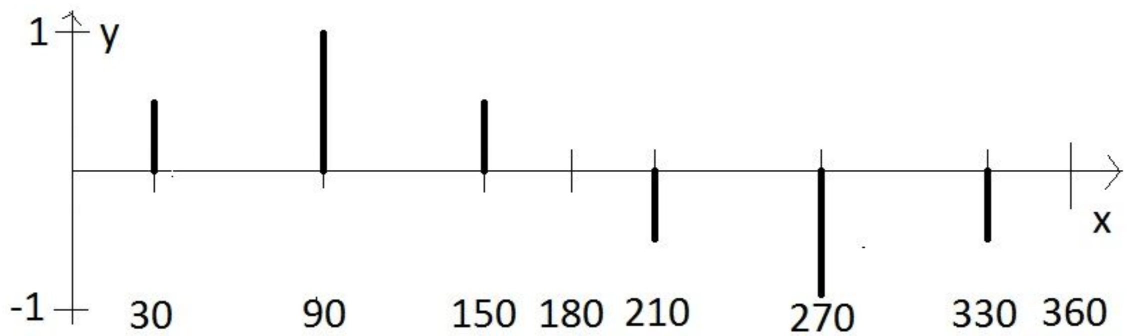
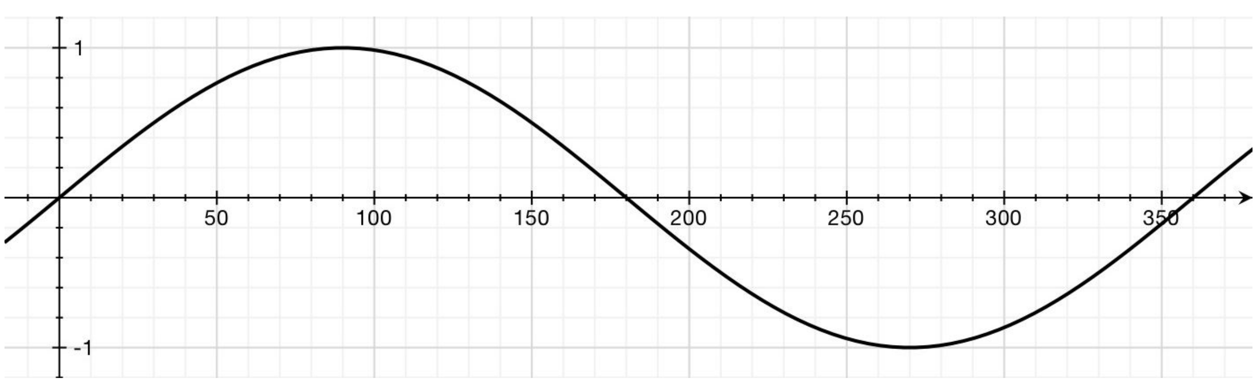


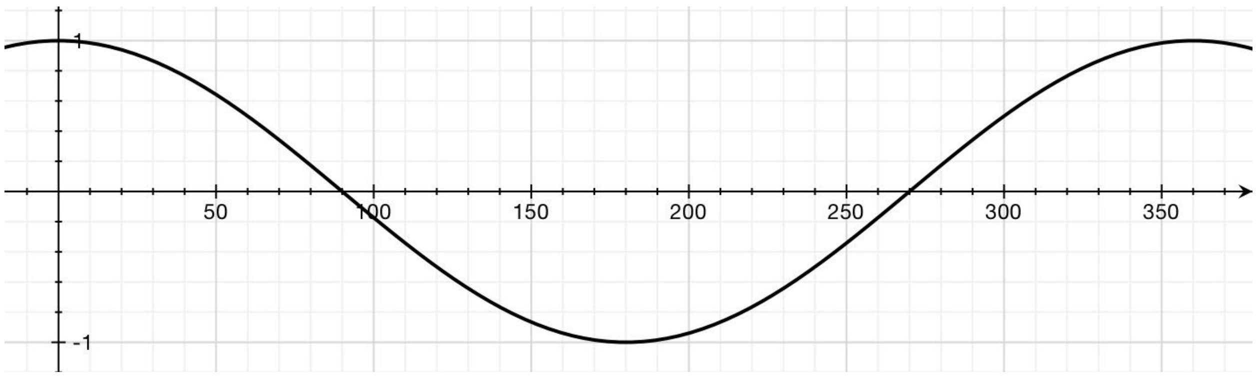
In each little circle, the vertical line segment represents the value of sine at that point. If you put those little line segments onto a coordinate plane,you get a sketch that shows the general trend of the sine function for one cycle.



This is a continuous graph of the sine function:



This is a graph of the cosine function:



And here’s tangent: