Completing the Square

1 Introduction

A trinomial looks like $ax^2 + bx + c$. A trinomial that is a perfect square can be factored into two identical factors; it looks like $(dx + e)(dx + e)$ or $(dx + e)^2$. When you are working with second powers, there are times when it is useful to see things in terms of squares. This lesson will show you how to find square perfect trinomials in quadratic expressions.

2 How to Create a Perfect Square Trinomial

First consider the case where the leading coefficient is 1. Then the trinomial looks like $x^2 + bx + c$. (Note that b or c or both may be 0.)

Example: Create a perfect square trinomial by adding something to, or subtracting something from, the expression $x^2 + 4x$.

Notice that b, the coefficient of x in the second term, is 4. Divide it by 2, square it, and add that term to $x^2 + 4x$.

In a perfect square trinomial like this one, in which the first term's coefficient is 1, the second term's coefficient divided by 2 and squared is equal to the third term.

In other words, in an expression $x^2 + bx$, add $(b/2)^2$ to form a perfect square. So for $s^2 + 4x$, the square trinomial is $x^2 + 4x + 4$, which is equal to $(x + 2)^2$. Then the equation $x^2 + 4x = 0$ becomes $x^2 + 4x + 4 = 4$, or $(x + 2)^2 = 4$.

How do you go from $x^2 + 4x + 4$ to $(x + 2)^2$? The number b, – that is, the number that multiplies x, – divided by 2, will become the second term of the binomial In this example, the x term's multiplier is 4. $4/2 = 2$, so the binomial that gets squared is $x + 2$.

It gets a little more complicated when the leading coefficient is something other than 1. In that case you have to factor the leading coefficient out. For example, to complete the square for $3x^2 + 6x = 0$:

$$3x^2 + 6x = 0$$
$$3(x^2 + 2x) = 0$$

Next, divide b (in this case 2) by 2, square it, and add the result to the expression.

Then: $3(x^2 + 2x + 1) = 3$

Notice that adding 1 to the trinomial means adding 3 to the left-hand side of the equation – because 1 gets multiplied by 3 – so 3 gets added to the right-hand side as well.

Now express the trinomial as a square. That is, divide the b term (2) by 2, add the result (1) to x (to get $(x + 1)$ in this case), and square the resulting binomial. The result is a replacement for the trinomial.

Then $3(x + 1)^2 = 3$. And that is the result you need.

Example: Complete the square for the equation x2-2x-8=0. Add 8 to each side. Then the equation is $x^2 - 2x = 8$. Find b, the number that multiplies x. In this case it's -2. Divide b by 2 and square the result: -2/22=1. Add that result to both sides of the equation: x2-2x+1=9. Write the trinomial as a square: x-12=9.

3 Why This Works

If the trinomial $x^2 + bx + c$ is square, then its factored form it looks like $(x + m)(x + m)$. Multiply that out to get $x^2 + 2mx + m2$.

Compare these two expressions for the same thing:

$$x^2 + 2mx + m^2$$
$$x^2 + bx + c$$

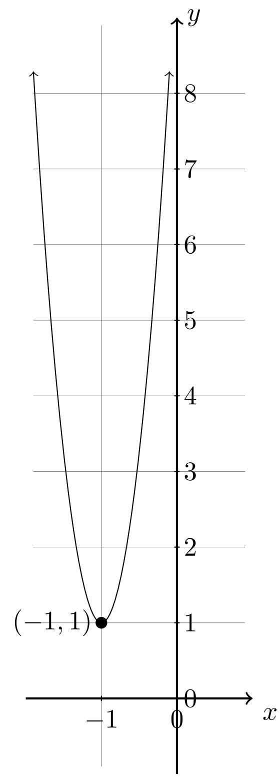
In the first expression, the square root of the last term (or possibly the negative square root), multiplied by 2, is the multiplier in the middle term. Since the two expressions represent the same thing, that means that in the second equation b is equal to the square root or the negative square root of c multiplied by 2. This is the case for every square trinomial whose leading coefficient is 1. For example in x^2+2x+1, which is a square trinomial (it factors to $(x+1)(x+1)$), the square root of the constant term is 1, and the multiplier in the middle term is $2\sqrt{1} = 2$.

When you complete the square you apply this reasoning in the opposite direction: You know b but not c, so you divide b by 2 and square the result to find the final term. For example, to find the number to add to $x^2 + 2x$ to form a square trinomial, divide 2 (b) by 2 and square it. $(2/2)2 = 1$. So c is 1 and the trinomial is $x^2 + 2x + 1$.

4 Applications

Example: a parabola. Rewrite the equation $9x^2 + 18x + 8 = 0$ in vertex format, that is, in the format $y = (x - h)^2 + k$.

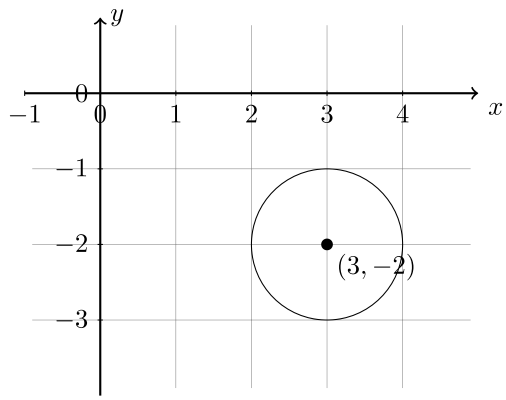
$y = 9x^2 + 18x + 8$	
$y = 9(x^2 + 2x) + 8$	Factor out 9 from the first two terms.
$x^2 + 2x = -8$	Subtract 8 from both sides.
$2/2 = 1$	Divide b (2 in this case) by 2.
$1^2 = 1$	Square the result.
	Add 1 inside the parentheses to complete the square.
$y = 9(x^2 + 2x + 1) - 8 + 9$	Subtract 9 outside the parentheses for a net change of 0.
$y = 9(x + 1)^2 + 1$	Factor and simplify.



Example: a circle. The equation $x^2 + y^2 - 6x + 4y + 12 = 0$ describes a circle. Rewrite the equation in standard format.

Standard format for the equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the circle's center and r is its radius. To convert something that looks like $x^2 + y^2 + ax + by + c = 0$ (the general form for the equation of a circle) into standard format, you need to complete a couple of squares, one in x and one in y.

$x^2 - 6x + y^2 + 4y + 12 = 0$	Put x terms with x terms and y terms with y terms.
$x^2 - 6x + y^2 + 4y = -12$	Subtract 12 from both sides to get it out of the way.
	Complete the square for x.
$x^2 - 6x+$	$x^2 - 6x+$ something is a perfect square.
$(-6/2)^2 = 9$	9 is the needed number.
$x^2 - 6x + 9$	$x^2 - 6x + 9$ is a perfect square.
$(x - 3)^2$	$x^2 - 6x + 9$ expressed as a square of a binomial.
$(x - 3)^2 = x^2 - 6x + 9$	9 was added to complete the square
	Complete the square for y.
$y^2 + 4y+$	$y2 + 4y+$ something is a perfect square.
$(4/2)^2 = 4$	4 is the needed number.
$y^2 + 4y + 4$	$y^2 + 4y + 4$ is a perfect square.
$(y + 2)^2$	$y^2 + 4y + 4$ expressed as a square of a binomial.
$(y + 2)^2 = y^2 + 4y + 4$	4 was added to complete the square
$x^2 - 6x + y^2 + 4y = -12$	
$x^2 - 6x + 9 + y^2 + 4y + 4 = -12 + 9 + 4$	Add to the right what you add to the left.
$(x - 3)^2 + (y + 2)^2 = -12 + 9 + 4$	
$(x - 3)^2 + (y + 2)^2 = 1$	Simplify.



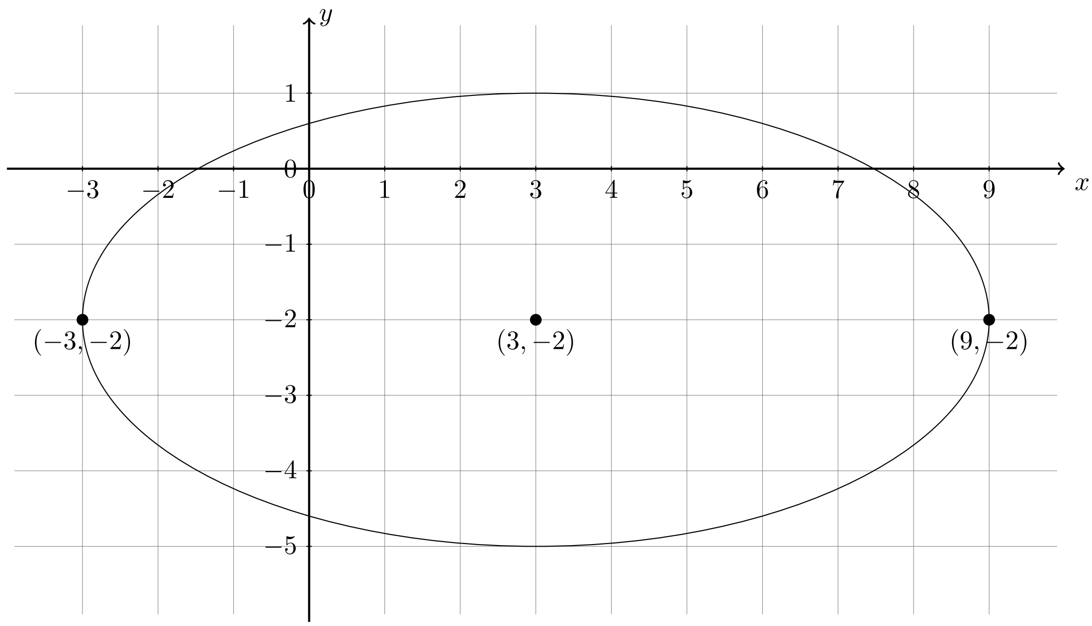
Example: an ellipse

The equation $x^2 + 4y^2 - 6x + 16y - 11 = 0$ describes an ellipse. Rewrite the equation in standard format.

The standard form of the equation of an ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where (h, k) is the location of the ellipse's center and a and b are the lengths of the ellipse's axes.

As in the case of the circle, you need to complete two squares here. It's a bit trickier here, because the square terms you start with may have multipliers. And in the end you have to make it all equal 1.

$x^2 - 6x + 4y^2 + 16y - 11 = 0$	Put x terms with x terms and y terms with y terms
$x^2 - 6x + 4y^2 + 16y = 11$	Add 11 to both sides.
	Complete the square for x.
$x^2 - 6x+$	$x^2 - 6x+$ something is a perfect square.
$(-6/2)^2 = 9$	9 is the needed number.
$x^2 - 6x + 9$	$x^2 - 6x + 9$ is a perfect square.
$(x-3)^2$	$x^2 - 6x + 9$ expressed as a square of a binomial.
$(x-3)^2 = x^2 - 6x + 9$	9 was added to complete the square
	Complete the square for y.
$4(y^2 + 4y)$	Factor out the common factor.
$y^2 + 4y+$	$y2 + 4y+$ something is a perfect square.
$(4/2)^2 = 4$	4 is the needed number.
$y^2 + 4y + 4$	$y^2 + 4y + 4$ is a perfect square.
$(y+2)^2$	$y^2 + 4y + 4$ expressed as a square of a binomial.
$4(y+2)^2 = 4y^2 + 16y + 16$	16 was added to complete the square
	Put it all together.
$x^2 - 6x + 9 + 4y^2 + 16y + 16 = 11 + 9 + 16$	Add to the right what was added to the left.
$(x-3)^2 + 4(y+2)^2 = 36$	
$\frac{(x-3)^2}{36} + \frac{(y+2)^2}{9} = 1$	Divide both sides by the number on the right, 36.



Uses: solving quadratics and deriving the quadratic formula; deriving the equations of conics.