Factor Theorem and Remainder Theorem

Factor Theorem

The factor theorem applies to polynomial functions and states that:

- If $x - c$, where c is some real number, is a factor of $f(x)$, then c is a solution to $f(x) = 0$ and
- If c is a solution to $f(x) = 0$, then $x - c$ is a factor of $f(x)$.

The factor theorem is really just an extension of the factoring method for solving quadratic equations. Let's apply it to a quadratic equation.

Example

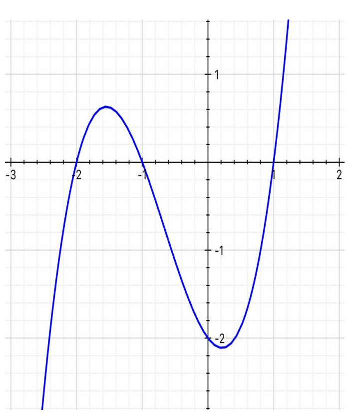
Solve the equation $x^2 + 3x + 2 = 0$ by factoring.

The factors of $x^2 + 3x + 2$ are $x + 1$ and $x + 2$. Their product can equal zero only if at least one of them equals zero. Therefore the equation $x^2 + 3x + 2 = 0$ has two zeros, one for each factor:

$x + 1 = 0$	$x + 2 = 0$
$x = -1$	$x = -2$

So the solution to is $x^2 + 3x + 2 = 0$ is $x = -1 \ and \ x = -2$.

The last step uses the first bullet of the remainder theorem as stated above. For example, in the case of $x + 1 = 0$, $c = -1$ and -1 is a solution to $x^2 + 3x + 2 = 0$.

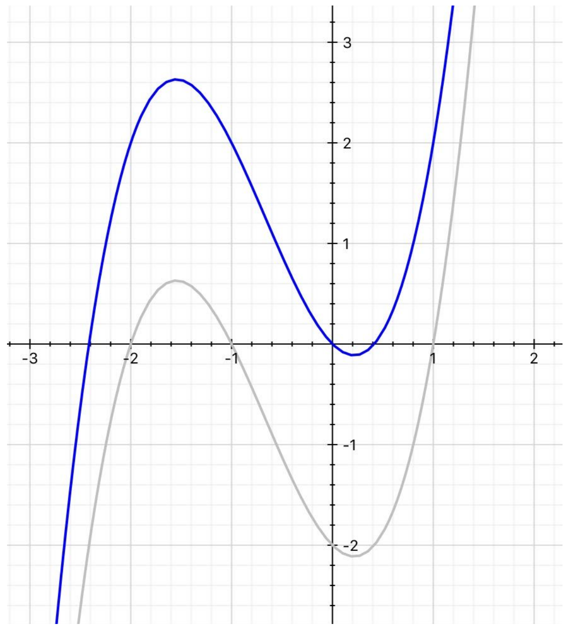


But not every polynomial is just a quadratic. Let's try this with a longer polynomial function: $g(x) = x^3 + 2x^2 - x - 2$. See its graph at left. From the graph, you can see that $y = 0$ where $x = -2, \ -1, \ and \ 1$, so the solutions to $g(x) = x^3 + 2x^2 - x - 2$ are $x = -2$, $x = -1$, and $x = 1$. And the factors of $x^3 + 2x^2 - x - 2$ are $x + 2$, $x + 1$, and $x - 1$. That illustrates the second bullet of the factor theorem as stated above: If c is a solution to $f(x) = 0$, then $x - c$ is a factor of $f(x)$.

Remainder Theorem

But not every polynomial can be factored. Consider the polynomial $h(x) = x^3 + 2x^2 - x$. That's a vertical shift of $g(x) = x^3 + 2x^2 - x - 2$ two units upward. $h(x)$ doesn't factor, so you can't solve it with the factor theorem. And those nice integer zeros that we had for $x^3 + 2x^2 - x - 2 = 0$ are no longer zeros. Since 2 has been added to the function's every y-value, those x-values that used to be zeros now have a y-value of 2.

The function $h(x) = x^3 + 2x^2 - x$ graphs like this:



The blue curve is the transformed function, $h(x) = x^3 + 2x^2 - x$, and the gray curve is the original function, $g(x) = x^3 + 2x^2 - x - 2$, thrown in for reference.

Our nice zeros from before are gone, but the x-values that had them now have nice y-values of 2. If you divide $h(x) = x^3 + 2x^2 - x$ by any of the factors of $g(x)$ -- -2, -1, or 1 -- you will get a remainder of 2. That's because $g(x)$ is divisible by those factors and the value of $h(x)$ at those points is 2 more.

The remainder theorem says that if you divide a polynomial function by a factor $x - c$ you will get a remainder that is equal to the value of the function at point c.

To use the remainder theorem to find the remainder when a polynomial is divided by a binomial $x - c$, evaluate the function at that point. Consider our example $h(x) = x^3 + 2x^2 - x$. To find the remainder when $h(x)$ is divided by $x - 1$, evaluate the function at $x = 1$. That's $1^3 + 2 \cdot 1^2 - 1 = 2$, so the remainder you are looking for is 2. Yes, you could have found that result by division, but then you would not have been using the remainder theorem.

To use the remainder theorem to find the value of a polynomial function at a point c, divide the polynomial by $x - c$. The remainder is the value of the function at point c. If the remainder is zero, the polynomial is divisible by $x - c$ and c is a zero of the function.