Logarithms

Introduction

Some background: inverse functions

An inverse function undoes a function. For example, subtraction undoes addition, so subtraction is the inverse function of addition. Division undoes multiplication, so division is the inverse function of multiplication.

A function takes x to y. You start with x and the function tells you y. An inverse function takes you from y to x. Consider the function $f(x) = x + 2$. If you input, say, 3 for x, the function takes you to 5 for y. The inverse function is $g(x) = x - 2$. Take the result that f gave you and input it into g and you get $g(5) = x - 2 = 3$: The result takes you back where you started from.

In inverse functions, x and y exchange roles. In the example above, for function f, $x = 3$ and $y = 5$; for the inverse function g, $x = 5$ and $y = 3$.

Consider the exponential function: $y = a^x$. What is its inverse function? What undoes it? That would be $x = a^y$. But a function needs to be expressed with y isolated on the left-hand side of the equation. That is what the log function does. $y = log_a x$ means the same thing as $x = a^y$.

Example: $100 = 10^2$. This is equivalent to $log_{10} 100 = 2$. In both equations, the number 10 is called the *base*. The exponent in the first equation is the log in the second.

The basic logarithmic relationship is this:

$y = log_a x$ means the same thing as $x = a^y$

or

$$y = log_a x \text{ if and only if } x = a^y.$$

Graph

Let's use the exponential function to get some ideas about the graph of the log function. Since the exponential and logarithmic functions are inverses, the domain of the log function is the range of the exponential function, and the range of the log function is the domain of the exponential function.

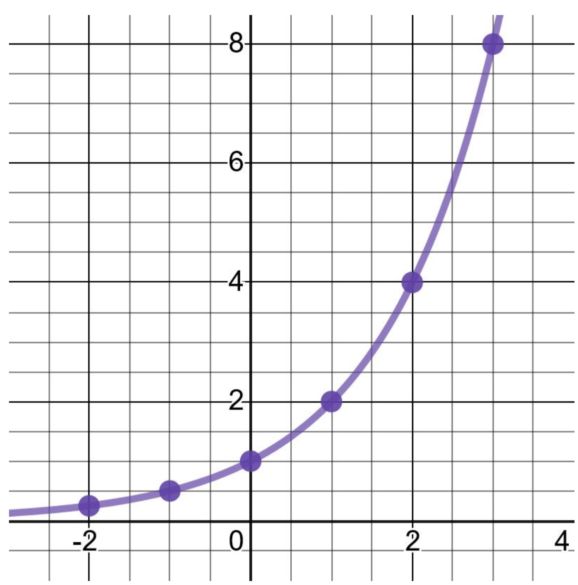
	Domain	Range
Exponential function	$(-\infty, \infty)$	$(0, \infty)$
Log function	$(0, \infty)$	$(-\infty, \infty)$

The log function's domain is positive real numbers and its range is all real numbers.

Inverse functions reflect each other in the line $y = x$. Let's graph up an exponential function, $y = 2^x$, and its corresponding log function $y = log_2 x$, and see how that looks.

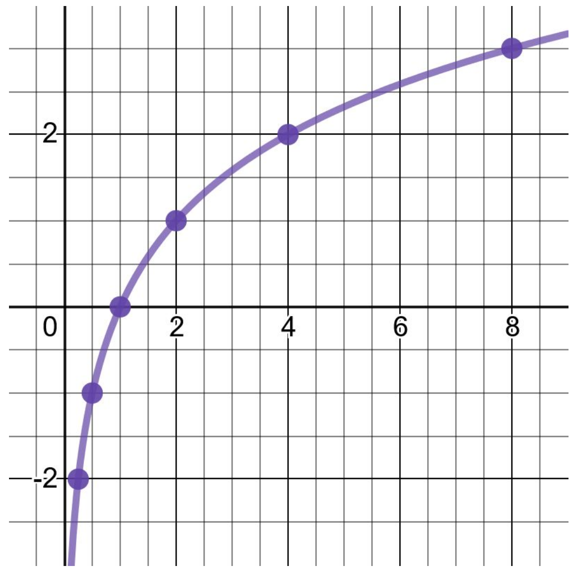
Let's start with the exponential function:

x	2^x
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8

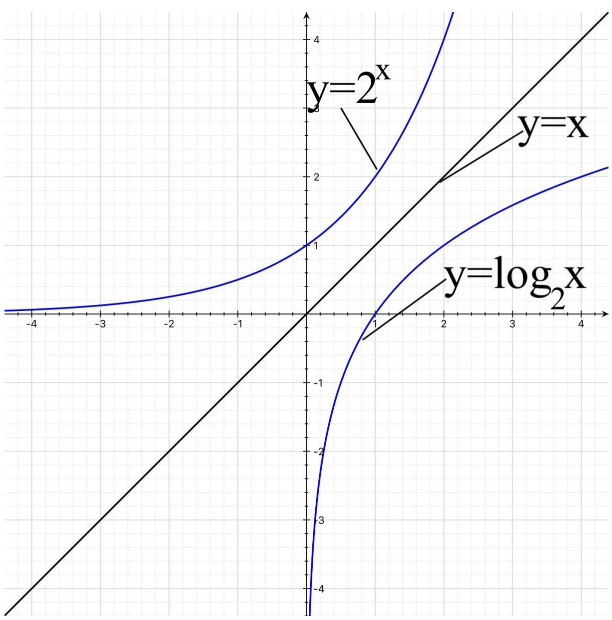


Now let's reverse those ordered pairs to get points on the logarithmic function and let's connect them into a curve:

x	$log_2 x$
1/4	-2
1/2	-1
1	0
2	1
4	2
8	3



On the same graph -- and with $y = x$ thrown in for reference -- it looks like this:



Log function characteristics:

- The exponential and log graphs reflect each other in the line $y = x$, as do every pair of inverse functions.

- The log function is asymptotic to the y-axis. Because the log function's domain is just positive numbers, its graph is located entirely to the right of the y-axis.
- The log function increases over all of its domain and there is no limit to how high it goes, but its rate of increase decreases as x increases.

What kind of number can be the base -- a in $log_a x$ -- for the logarithmic function? Once again, consider the exponential function. $y = a^x$ means the same thing as $log_a y = x$. In the exponential function, as you know, the base, a, can be any positive real number. The same constraint applies to the logarithmic function. A log base can be any positive real number. It need not be rational.

Log Bases

Two logarithm bases that are used frequently are 10 and e.

Base 10 is easy to work with because our number system is base 10. For example, $log_{10} 100 = 2$ (because $10^2 = 100$) and $log_{10} 1000 = 3$ (because $10^3 = 1000$). Log_{10} is typically written just as "log," not "log_{10}," and is called a *common logarithm.*

The other base you will see often is e. What is e?

Consider the expression $(1 + 1/n)^n$. Evaluate it for various values of n:

n	$(1 + 1/n)^n$
1	2
2	9/4=2.25
3	$64/27 \approx 2.37$
4	$625/256 \approx 2.44$
5	$7776/3125 \approx 2.49$

As n gets larger, $(1 + 1/n)^n$ also gets larger, but only up to a point, a *limit*. That limit is called e. It is an irrational number. That means it cannot be expressed as a quotient of integers -- so its exact value cannot be expressed in decimal form. It is equal to approximately 2.7183. A logarithm base e is called a *natural logarithm.* The natural log of x is written $ln(x)$.

Base e comes up often with exponential growth and decay, as well as with continuous interest.

Examples

1. Express $ln(4) = x$ in exponential form.

   Consider the basic logarithmic relationship: $log_a b = c \Leftrightarrow a^c = b$. Remember that ln means log_e. So you can express $ln(4) = x$ as $log_e(4) = x$ and then use the basic logarithmic equation to say $log_e 4 = x \Leftrightarrow e^x = 4$. (Note: The expression log_e is not considered good form, so be careful not to write it in an answer.) Answer: $e^x = 4$

2. Simplify: $lne = x$.

   Again consider that $log_a b = c \Leftrightarrow a^c = b$. If $log_e e = x$, then $e^x = e$, so $x = 1$. Answer: $x = 1$.

3. Simplify $ln(e^2) = x$.

   $log_e e^2 = x$

   $log_e e^2 = x$

   $e^2 = e^x$

   $e^x = e^2$

   Both the exponential function and the logarithmic function are one to one, meaning that for every y there is only one x. For the exponential function, that means that if $a^b = a^c$, then $b = c$. So if $e^x = e^2$ then $x = 2$.

   Answer: $x = 2$

4. Simplify $log_3(1/9)$

$log_3(1/9) = x$	Give a letter name to the thing you are looking for.
$3^x = 1/9$	Use the basic relationship, $log_a b = c \Leftrightarrow a^c = b$, to convert this logarithmic equation into exponential form.

$3^x = 3^{-2}$	Convert the right-hand side into exponential form.
$x = -2$	Use the one-to-one property of the exponential function.

Answer: $x = -2$

Another way to approach the problem is to say $log_3(1/9)$ is the power to which you take 3 to get $1/9$. That is -2, so $log_3(1/9) = -2.$ You may want to use one method to solve the problem and another to check.

5. Simplify $ln(\sqrt{e})$

A format change will help. $\sqrt{e} = e^{1/2},$ so $ln(\sqrt{e}) = ln(e^{1/2})$. Now give that thing a name: $ln(e^{1/2}) = x$ or $log_e(e^{1/2}) = x.$ Apply the basic relationship:

$$\log_e e^{1/2} = x$$
$$e^{1/2} = e^x$$

Then $x = 1/2$

Answer: $ln(\sqrt{e}) = 1/2$

6. Simplify $log_{\sqrt{5}}(25)$.

Use a letter to name the thing you are looking for: $log_{\sqrt{5}}(25) = x.$

Apply the basic relationship to make the equation exponential: $\sqrt{5}^x = 25.$

What power of $\sqrt{5}$ equals 25? $(\sqrt{5})^2 = 5$ and $5^2 = 25,$ so $((\sqrt{5})^2)^2 = \sqrt{5}^4 = 25.$ So if $\sqrt{5}^x = 25,$ then $x = 4.$

Answer: $log_{\sqrt{5}}25 = 4.$

7. Let $f(x) = ln(x-1).$ Find the domain of function $f.$

Remember that the domain of the log function is positive real numbers. That means the argument of function function f -- the expression in parentheses -- has to be a positive number. Set up an inequality to solve for x :

$$x - 1 > 0$$
$$x > 1$$

In interval notation, that's $(1, \infty)$.

Answer: $(1, \infty)$

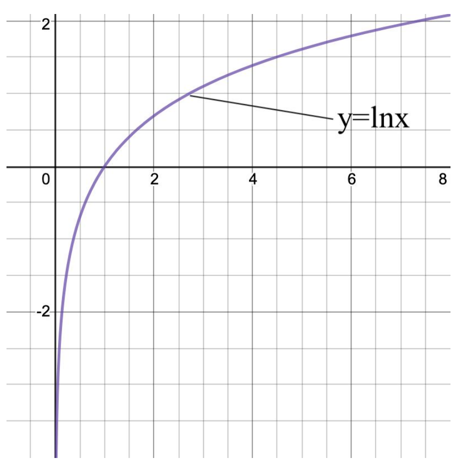
8. Let $f(x) = \sqrt{ln(x)}$. Find the domain of function f.

Because you can take the square root of any number that is at least 0, the x-values for which $ln(x)$ is greater than or equal to 0 comprise the domain of the function $f(x) = \sqrt{ln(x)}$. That means $ln(x)$ must be at least 0:

$$ln(x) \geq 0$$

Where is the ln function positive? Where is it zero?

Consider the graph of $ln(x)$.

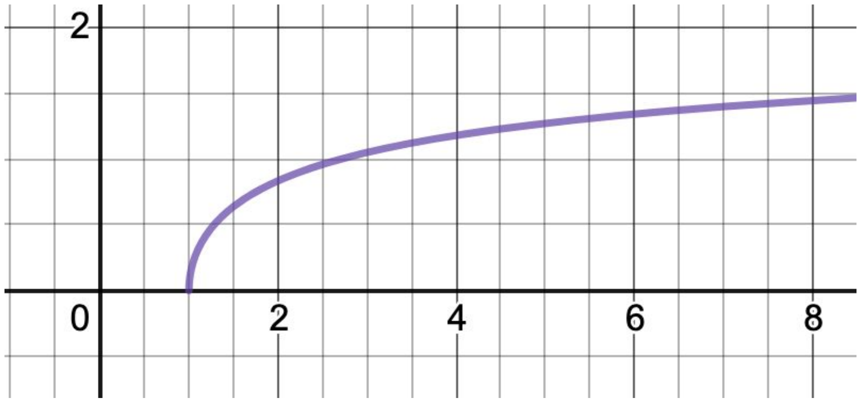


$ln(1) = 0$ and the natural log of anything greater than 1 is positive.

So the function $f(x) = ln(x)$ is greater than or equal to 0 for $x \geq 1$ or for x on the domain $[1, \infty)$.

Answer: The domain of $f(x) = \sqrt{ln(x)}$ is $[1, \infty)$.

Here's a graph of $f(x) = \sqrt{ln(x)}$:



9. Find $log(100, 000, 000)$.

   Remember that “log” with no base specified means log_{10}.

   Write $100, 000, 000$ in exponential form: $100, 000, 000 = 10^8$.

   Apply the basic logarithmic relationship: $y = log_a x$ means the same thing as $x = a^y$:
   $100, 000, 000 = 10^8$ means the same thing as $log_{10} 100, 000 = 8$.

   Answer: $log\ 100, 000 = 8$

   Note that “log(something)” is the power to which 10 is taken to get that something. This makes log_{10} convenient to work with when you are dealing with orders of magnitude -- that is, powers of 10.

10. Simplify $log 10^{10}$.

    Remember that “log” with no base specified means log_{10}.

    Use a letter to name the thing you are looking for: $log 10^{10} = z$.

    Use the basic logarithmic relationship -- $y = log_a x$ means the same thing as $x = a^y$
    -- to restate the equation: $z = log_{10} 10^{10}$ means the same thing as $10^{10} = 10^z$. Then by the one-to-one property of logarithms, $z = 10$.

    Another way to look at it: To what power (log) do you need to take 10 to get 10^{10}? Well, of course, 10.

Answer: $log10^{10} = 10$

Properties of Logarithms

Property $log_a 1 = 0$

By the basic logarithmic relationship, the statement $log_a 1 = 0$ is equivalent to $a^0 = 1$. And that statement is true for any real value of a except 0. The log base, a in this case, is always positive and therefore never 0, so for all possible log bases $log_a 1 = 0$.

Example: What is $ln 1$?

Log base anything of 1 equals 0. "Ln" means log base e. So $ln 1$ -- $log_e 1$ -- equals 0.

Property $log_a a = 1.$

Let's explore this. Let's give a name to $log_a a$: $log_a a = b$. Now apply the basic logarithmic relationship: If $log_a a = b$ then $a^b = a$. If a can be any positive, real number, then the only power to which you can take a and get a as a result is 1, so $b = 1$ and $log_a a = 1$.

Example: Find $log_{10} 10$.

$log_{10} 10$ is just $log_a a$ with $a = 10$. So $log_{10} 10 = 1$.

Property $log_a a^r = r$

To explore this, give a different name to $log_a a^r$ and then solve for that value:

$log_a a^r = b$

Now apply the basic logarithmic relationship: $log_a a^r = b$ if and only if $a^b = a^r$. Then by the one-to-one property of logarithms, $b = r$ and $log_a a^r = r$.

Another way to look at it: To what power do you take a to get a^r? r, of course. So r is $log_a a^r$.

Example: Find $log_2 2^3$.

Since $log_a a^r = r$, $log_2 2^3 = 3$.

Property $a^{log_a M} = M$

Try the same strategy as above: Give a different name to the term on the left-hand side of the equation and then use the basic logarithmic relationship to solve for that value:

$$a^{log_a M} = b$$

If $a^{log_a M} = b$ then $log_a b = log_a M$.
Then by the one-to-one property of logarithms, $b = M$ and $a^{log_a M} = M$.

Example: Simplify $2^{log_2 4}$.

Since $a^{log_a M} = M$, $2^{log_2 4} = 4$.

Property $log_a (MN) = log_a M + log_a N$
This is the *product rule* for logarithms. Let's explore it.

Let's say $a^b = M$ and $a^c = N$. Then $log_a M = b$ and $log_a N = c$. What does $log_a (MN)$ look like? Well, $MN = a^b \cdot a^c = a^{b+c}$, so $log_a (MN) = b + c = log_a M + log_a N$.

Example: Find $log_{10} (100 \cdot 10,000)$.
$log_{10} (100) = 2$ and $log_{10} (10,000) = 4$, so by the product rule $log_{10} (100 \cdot 10,000) = 2 + 4 = 6$. To verify: $100 \cdot 10,000 = 1,000,000$ and $log_{10} (1,000,000) = 6$.

Property $log_a M^r = r log_a M$
This is the *power rule* for logarithms.

Why does it work this way? Think of $log_a M^2$. That's equal to $log_a (M \cdot M.)$ By the product rule, that's equal to $log_a M + log_a M$, and that is equal to $2 log_a M$. That shows that the power rule works when $r = 2$.

What if r is some number greater than 2? Think of $log_a M^n$. That's equal to $log_a (M \cdot M \cdot M \ldots)$ until there are n M s. By the product rule, that's equal to $n log_a M$. That shows that the power rule works when $r = n$.

Example: Express $log 10^6$ without an exponent.

How does this work?

The rule is:

$$\log_a M^r = r\log_a M$$

Applying it here gives:

$$\log_{10} 10^6 = 6\log_{10} 10$$

And as established above $log_{10}10 = 1$, so $log10^6 = 6 \cdot 1 = 6$.

Property $log_a\left(\frac{M}{N}\right) = log_a M - log_a N$

This is the quotient rule for logarithms.

Why does it work? $log_a\left(\frac{M}{N}\right) = log_a\left(MN^{-1}\right)$. By the product rule, that's $log_a M + log_a N^{-1}$.

And by the power rule, $log_a N^{-1} = -1 log_a N$, so $log_a\left(\frac{M}{N}\right) = log_a M - log_a N^{-1}$.

Example: Write $log_3 \frac{7}{2}$ as a difference.

$$log_3 \frac{7}{2} = log_3 7 - log_3 2$$

Example: Write the following expression as a sum and/or difference of logarithms. Express powers as factors:

$$ln\left(\frac{2x+3}{x^2-3x+2}\right)^2$$

Factor the rational expression's denominator: $x^2 - 3x + 2 = (x-1)(x-2)$

So $ln\left(\frac{2x+3}{x^2-3x+2}\right)^2 = ln\left(\frac{2x+3}{(x-1)(x-2)}\right)^2$

Use the power rule to make the exponent into a coefficient:

$$ln\left(\frac{2x+3}{(x-1)(x-2)}\right)^2 = 2ln\left(\frac{2x+3}{(x-1)(x-2)}\right)$$

Apply the quotient rule:

$2ln(\frac{2x+3}{(x-1)(x-2)}) = 2[ln(2x+3) - ln(x-1) - ln(x-2)]$

Note that both $ln(x-1)$ and $ln(x-2)$ get **subtracted.** That is because both came from the denominator.

Property $log_a M = \frac{log_b M}{log_b a}$

This is the *change-of-base formula.*

Why does it work?

Let's give a name to $log_a M$: Call it n. Then $log_a M = n$.

Then $a^n = M$ by the basic logarithmic relationship.

And $log_b a^n = log_b M$ by the one-to-one property of logarithms.

$nlog_b a = log_b M$ by the power rule.

Solve for n and you get $n = \frac{log_b M}{log_b a}$

And we said $log_a M = n$, so $log_a M = \frac{log_b M}{log_b a}$

How can it help?

Sometimes you need a logarithm to be a base that is different from the base in which it is given to you. Typically you need to get numbers into either ln or log format, because so many calculators can interpret only those two bases.

For example, use the LOG function on your calculator to find express $log_2 8$.

Remember that log with no base specified means log_{10}.

By the change-of-base formula, $log_2 8 = \frac{log_{10} 8}{log_{10} 2}$. The calculator will tell you that $\frac{log_{10} 8}{log_{10} 2} = 3.$

To check, let's find $log_2 8$ by hand.

$log_2 8 = c$. Give it a name.

$2^c = 8$ Apply the basic logarithmic relationship.

$c = 3$. 3 is the only power to which you can take 2 and get 8. Check.

You won't usually be able to check without a calculator. It's just that this example works well that way.

For a summary of logarithmic properties, go to this page and click on "reference sheet."