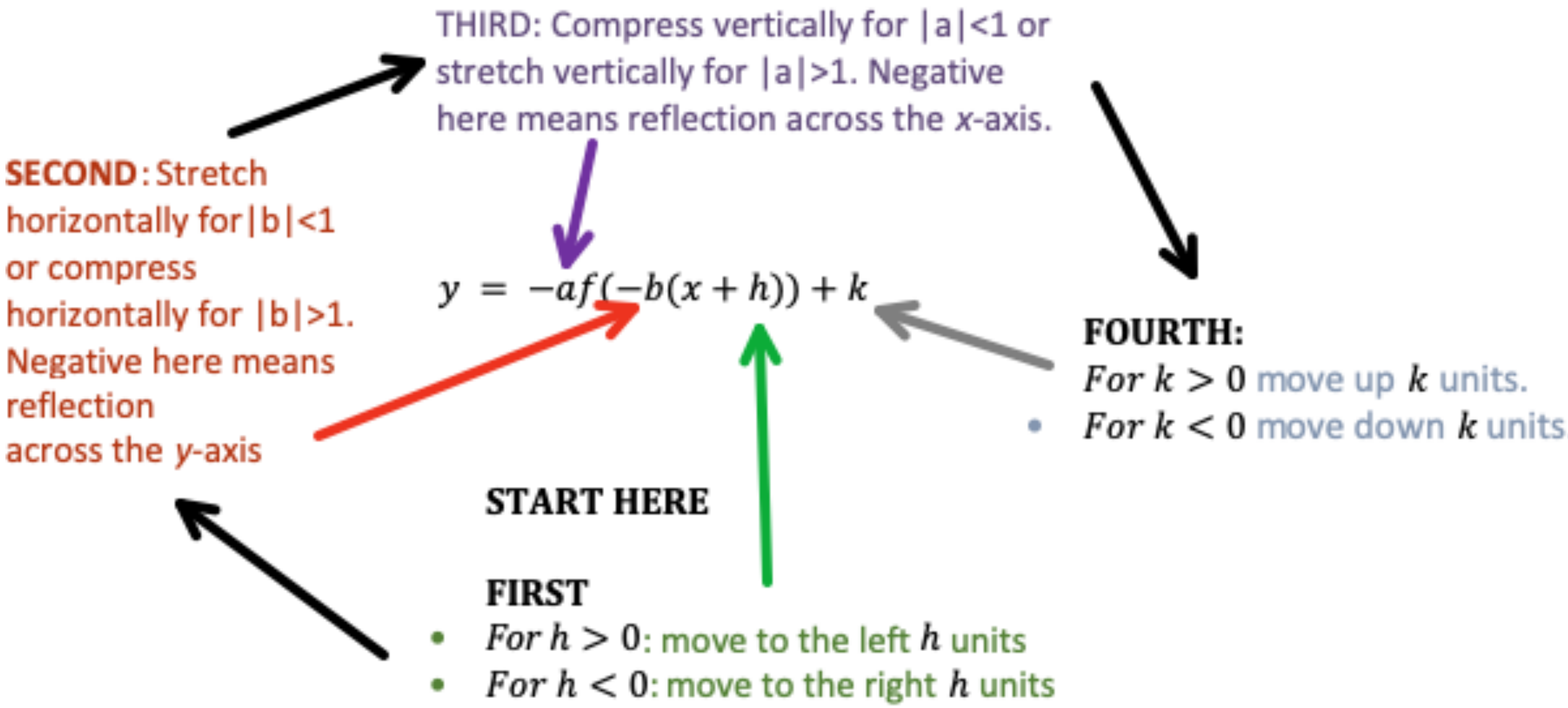
Transformation Rules for Functions

A Single Transformation

Transforming equation $y = f(x)$

Equation	How to obtain the graph
$y = f(x) + k \ \ (k > 0)$	Shift graph $y = f(x)$ up k units.
$y = f(x) + k \ (k < 0)$	Shift graph $y = f(x)$ down k units.
$y = f(x - h) \ \ (h > 0)$	Shift graph $y = f(x)$ right h units.
$y = f(x - h) \ (h < 0)$	Shift graph $y = f(x)$ left h units.
$y = -f(x)$	Reflect graph $y = f(x)$ over x-axis.
$y = f(-x)$	Reflect graph $y = f(x)$ over y-axis.
$y = af(x) \ \ (a > 1)$	Stretch graph $y = f(x)$ vertically by factor of a. (Multiply y -coordinates of $y = f(x)$ by a.)
$y = af(x) \ \ (0 < a < 1)$	Shrink graph $y = f(x)$ vertically by factor of a. (Multiply y -coordinates of $y = f(x)$ by a.)
$y = f(bx) \ \ (b > 1)$	Shrink graph $y = f(x)$ horizontally by factor of $1/b$. (Divide x -coordinates of $y = f(x)$ by b.)
$y = f(bx) \ \ (0 < b < 1)$	Stretch graph $y = f(x)$ horizontally by factor of $1/b$. (Divide x -coordinates of $y = f(x)$ by b.)

Putting it together



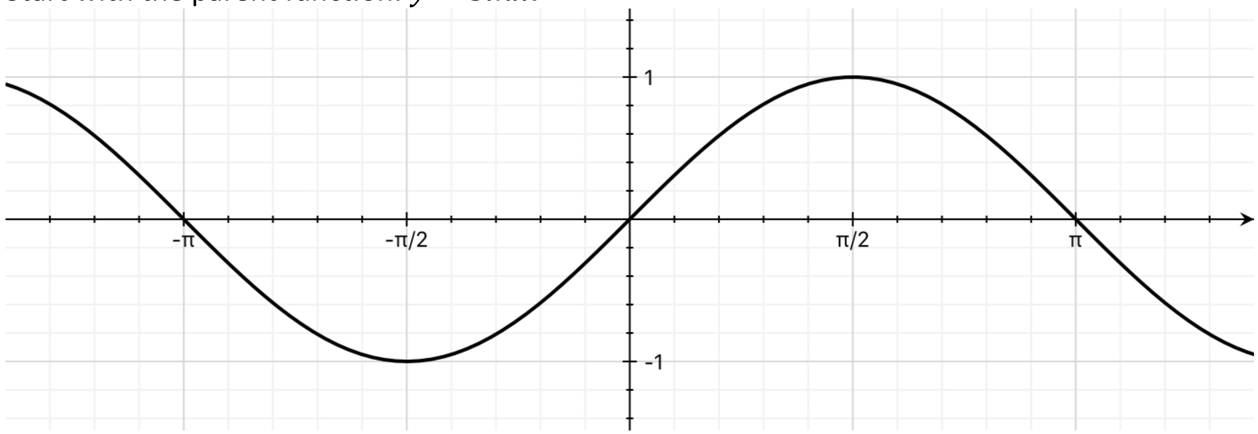
A Series of Transformations

Combining transformations can be tricky, because the order in which you carry them out may matter. (There are times when it does not make a difference -- and finding those situations can lull you into complacency.) Remember that a combination of transformations is a series of things that are being done to x. If you double a number and then add 3, you will most likely get a different result than you will if you add 3 to a number and then double it.

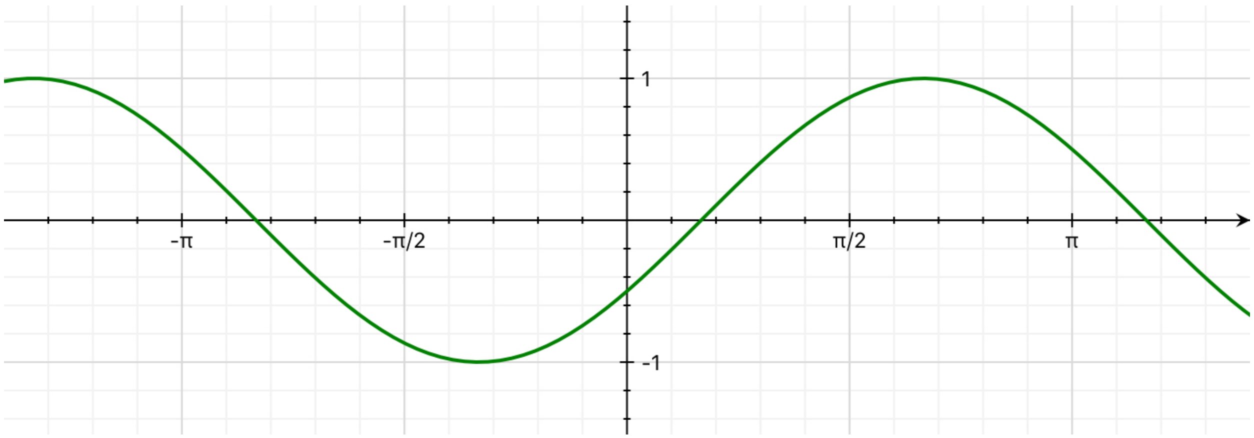
What is the right sequence? Start close to x and work your way out.

Example: $f(x) = 3sin(2(x - \pi/6)) + 1.$

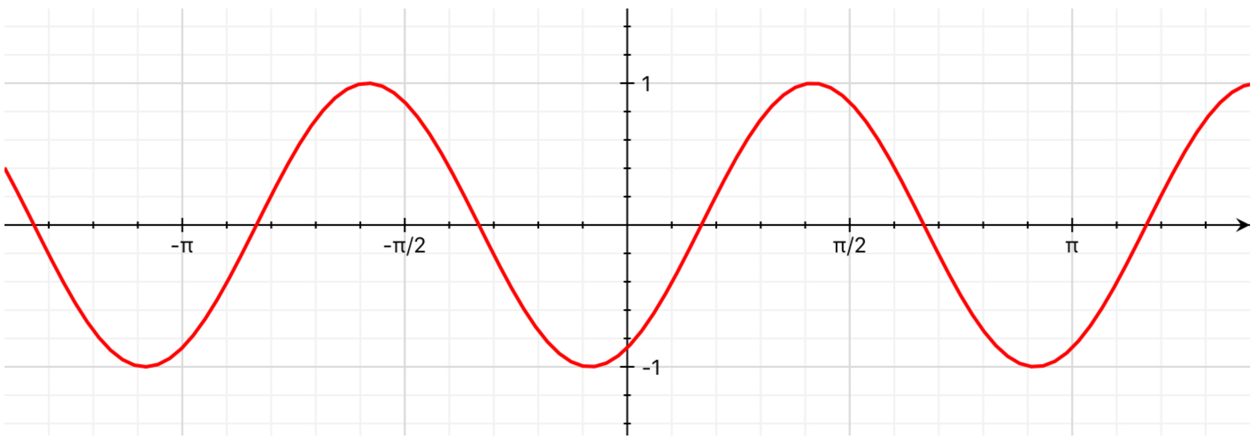
1. Start with the parent function: $y = sinx$.



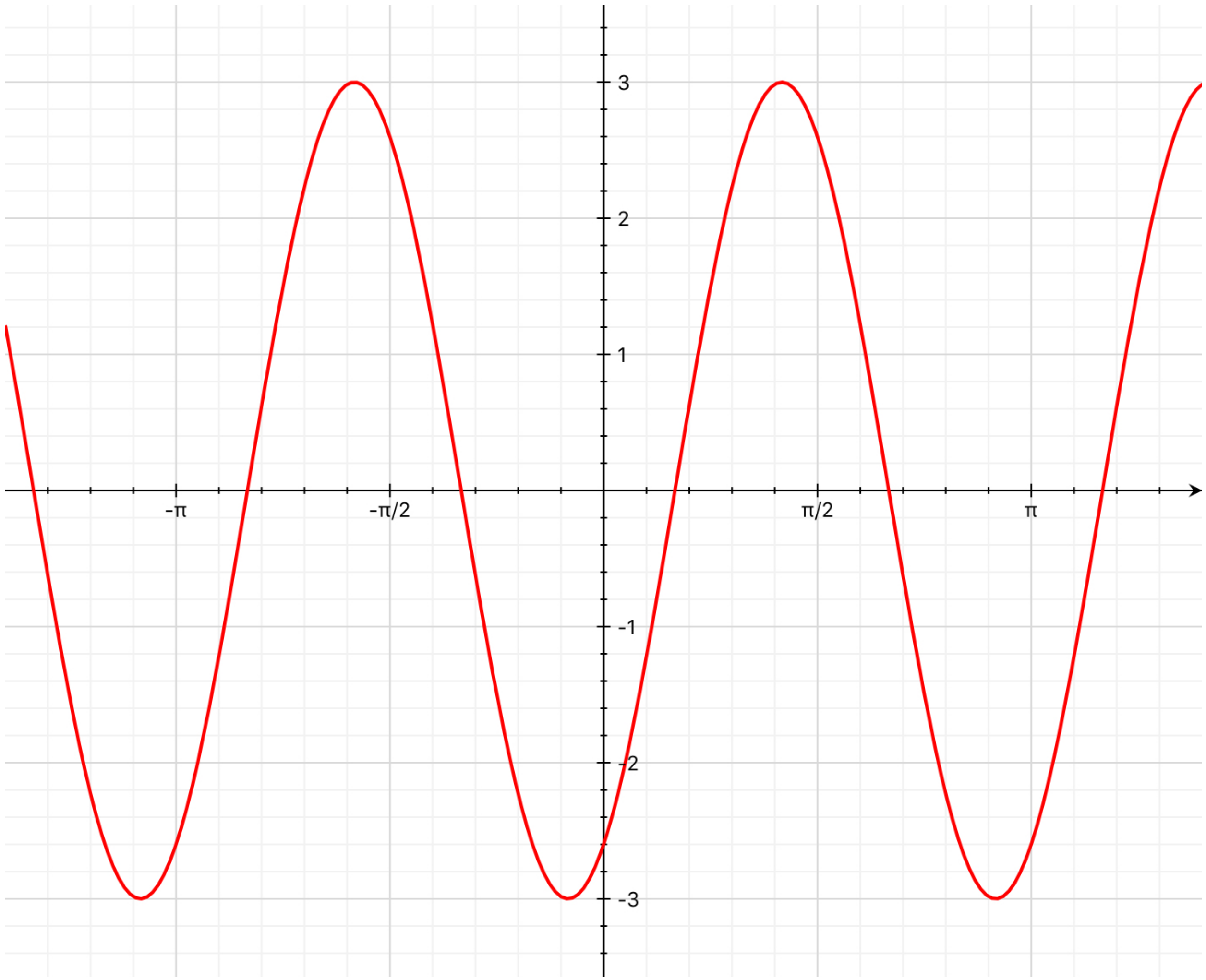
2. What's the first thing you do to x? Subtract $\frac{\pi}{6}$ from it. So the first thing you do to the graph is to move it $\frac{\pi}{6}$ to the right and get the graph of $y = sin\left(x - \frac{\pi}{6}\right)$



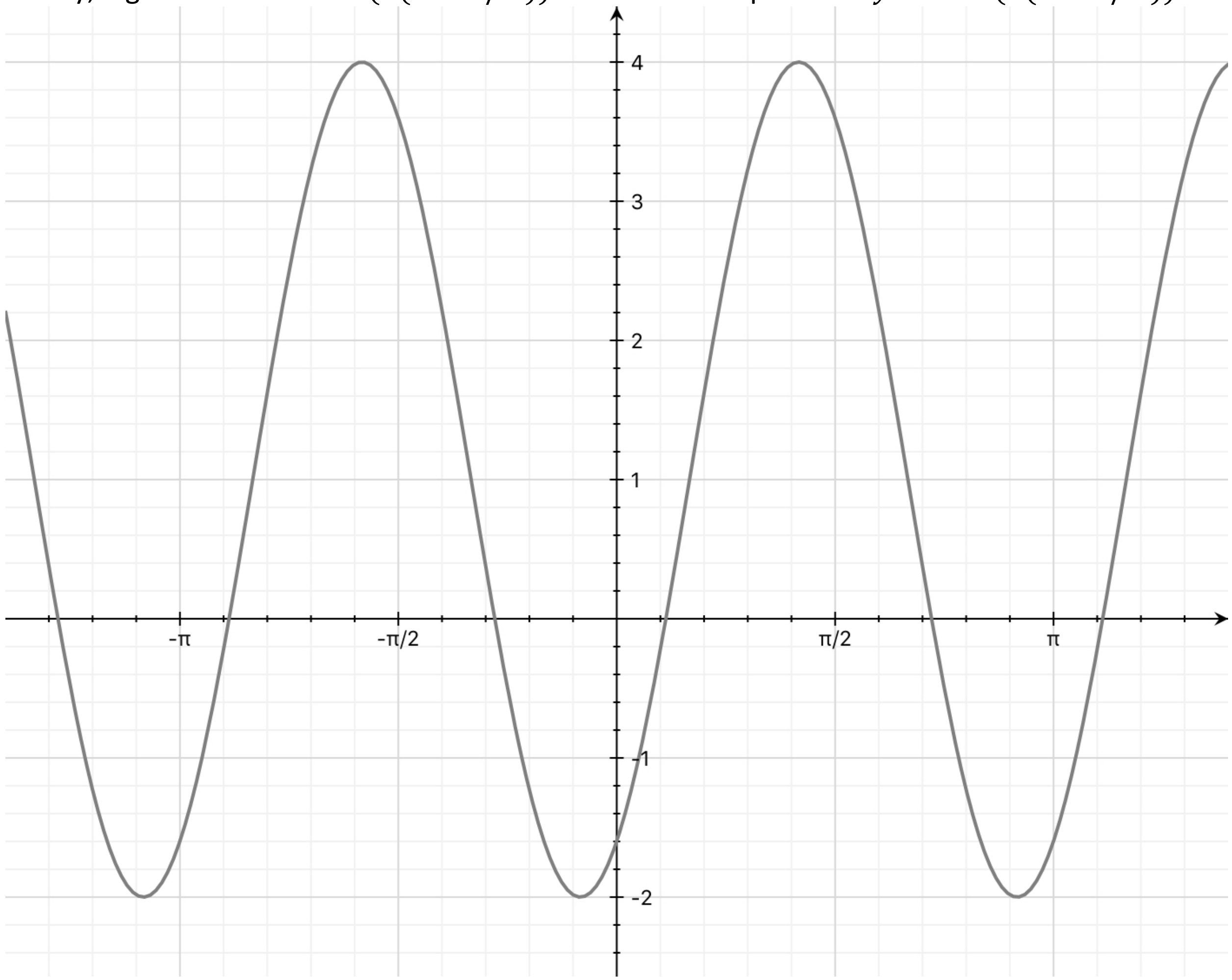
3. Next, multiply $x - \pi/6$ by 2 to get $y = sin\left(2(x - \pi/6)\right)$. On the graph that's a horizontal compression or, equivalently, moving along the x-axis twice as fast.



4. Next, $sin(2(x - \pi/6\ ))$ gets multiplied by 3. This stretches the graph vertically to three times its original height.


3
2
1
-1
-2
-3
-π
-π/2
π/2
π

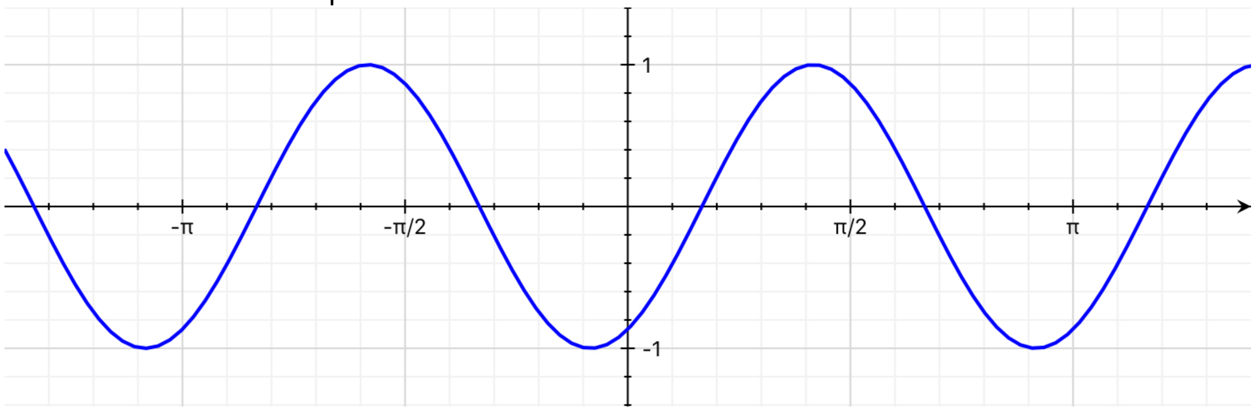
5. Finally, 1 gets added to $3sin(2(x - \pi/6))$ and the final equation is $y = 3sin(2(x - \pi/6)) + 1$. This raises the graph 1 unit.



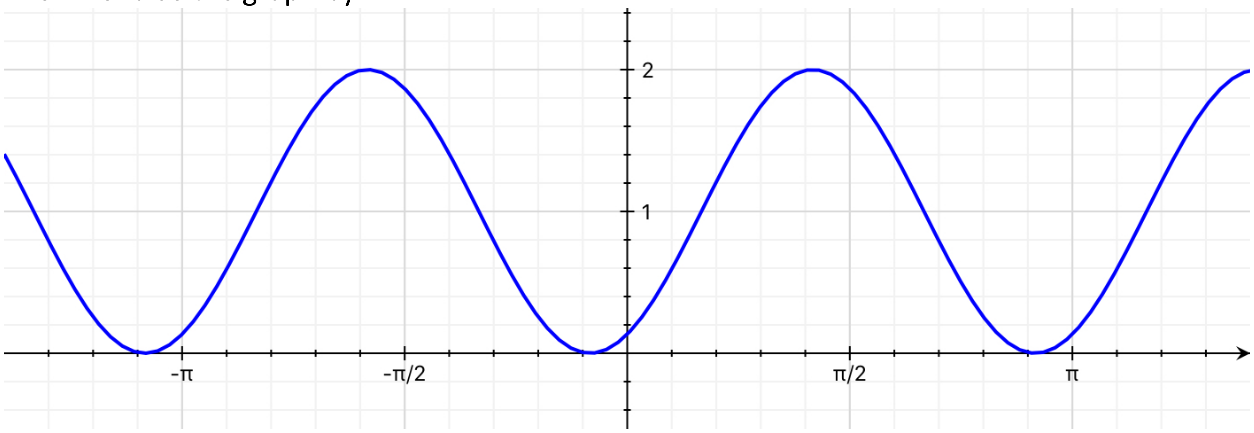
Discussion

Did I mention that the sequence in which you do the steps can be important? Let's consider a couple of places where it is:

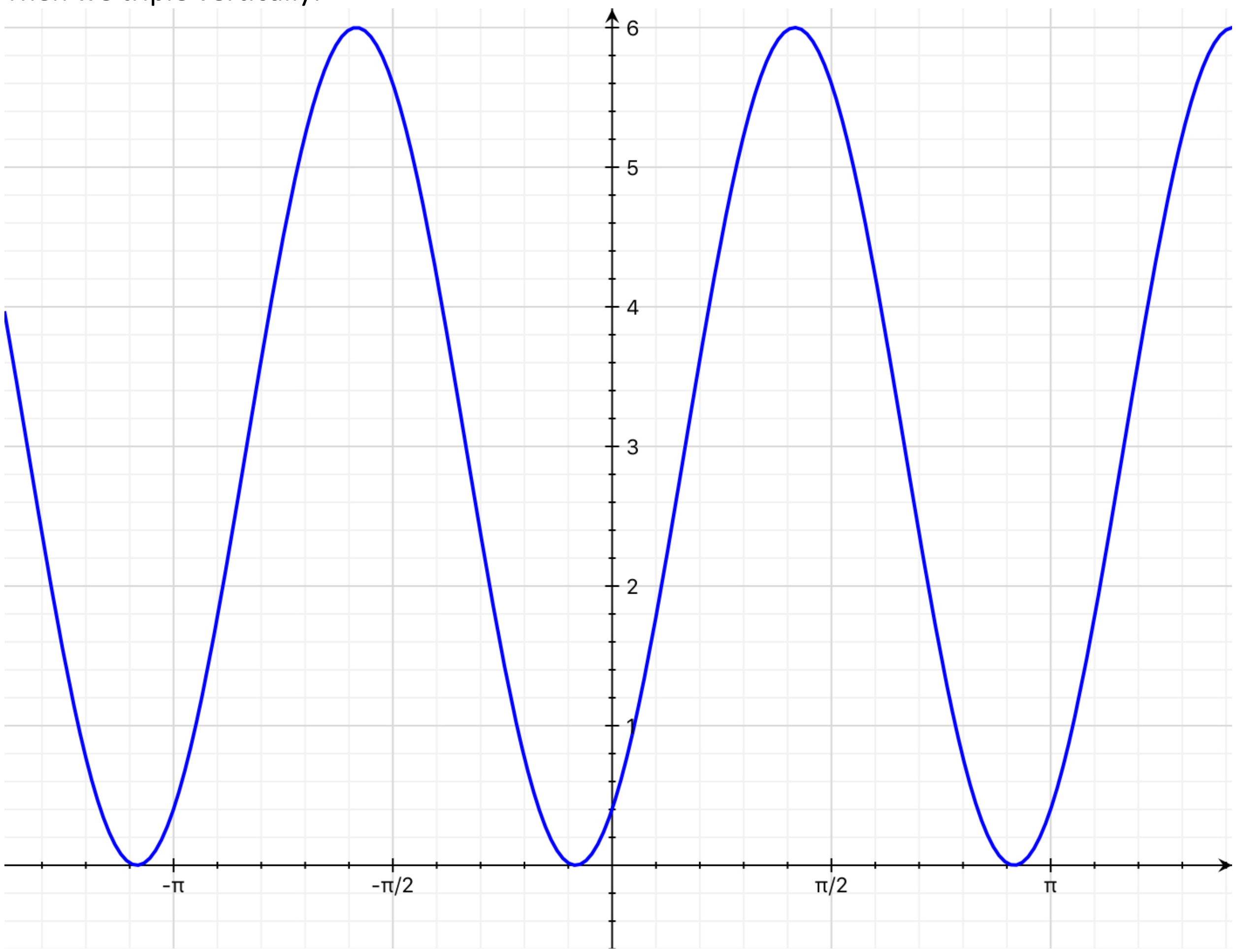
- Step 4 is a vertical tripling and step 5 raises the whole graph by 1. What if we first raise the whole graph by 1 and then triple vertically?
  - Where we were after step 3:



  - Then we raise the graph by 1:



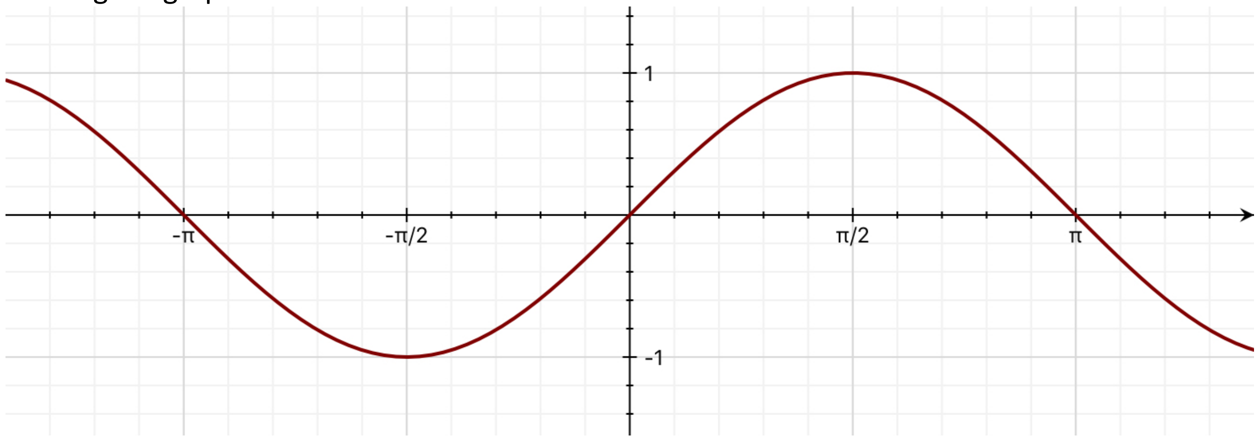
- Then we triple vertically:



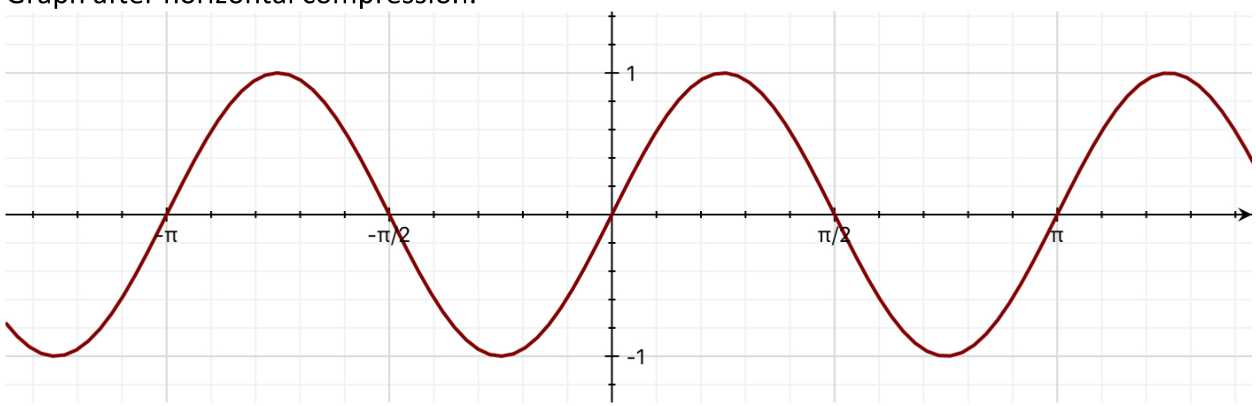
Compare that with the result we got when we tripled vertically first and then added 1. Different. When we added 1 before tripling, we included the 1 in the tripling, so we tripled higher numbers -- and ended up with higher numbers. Tripling after adding increased what was added.

- Another example: Step 2 is a horizontal shift and step 3 is a horizontal compression. What if we do the compression first and then the shift?

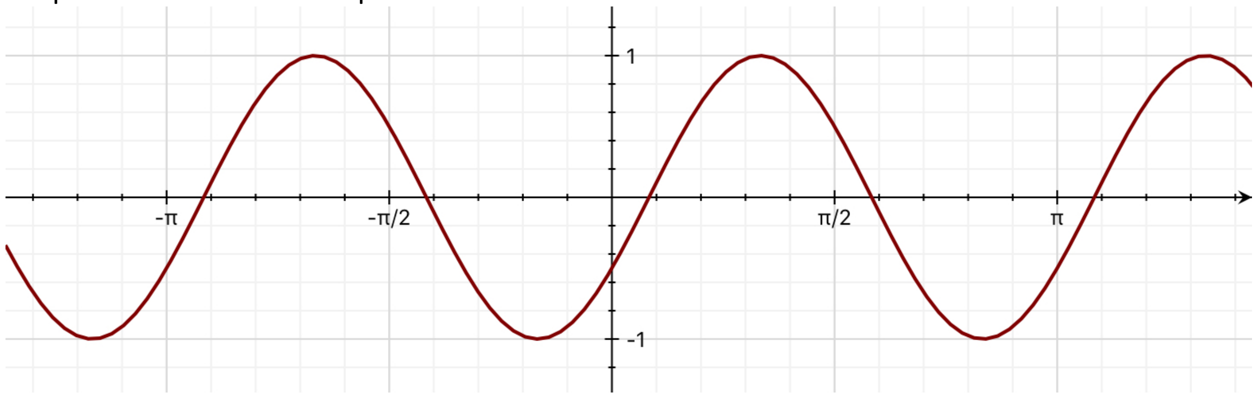
- Original graph:



- Graph after horizontal compression:



- Graph after horizontal compression and then horizontal shift:



If sequence didn't matter, this graph would look the same as step 3 in the original sequence. But it does not; it is shifted (slightly) to the left.

Does This Make Sense?

Students sometimes feel that some aspects of transformations seem backwards. $f(x-h)$ is h units to the *right* of $f(x)$, even though you are subtracting h -- shouldn't subtraction move it left? And $g(2x)$ is compressed from $g(x)$. Shouldn't multiplying by 2 stretch it out?

Horizontal translation   Let's say $y = x + 1$. This graph will be 1 unit higher than $y = x$, because we are adding 1 to every y-value. But what happens if we solve for x: $x = y - 1$? Now it looks like adding 1 to $y = x$ to make it $y = x + 1$ means making each x-value one less and thus moving it one unit to the left.

Another way to look at this: Consider $f(x) = x^2$ and $g(x) = (x-1)^2$. Let's look at some values:

x	x^2	$x-1$	$(x-1)^2$
-2	4	-3	9
-1	1	-2	4
0	0	-1	1
1	1	0	0

2	4	1	1

At $x = -1$, $(x-1)^2$ takes the value that $x - 1$ takes at $x = -2$. In fact, at every x value, $(x-1)^2$ takes the value that x takes one unit earlier. That puts $y = (x-1)^2$ one unit ahead of $y = x^2$. So subtracting a number from x moves the graph ahead.

Horizontal compression and stretching. Consider the same function, $f(x) = x^2$. Let's say $g(x) = (2x)^2$.

x	$f(x) = x^2$	$2x$	$g(x) = (2x)^2$
-4	16		
-3	9		
-2	4	-4	16
-1	1	-2	4
0	0	0	0
1	1	2	4
2	4	4	16
3	9		
4	`16		

Notice that $f(2) = g(1)$ and $f(4) = g(2)$. This pattern continues: f of a number equals g of twice that number. In other words, g increases twice as fast as f -- so g moves along twice as fast and g's graph is compressed compared with f's.