Practice Calculus Test with Trig

The problems here are similar to those on the practice test. Slight changes have been made.

1. Find the period of the function $f(x) = tan(2x + 1)$.

   The period of the tangent function, $g(x) = tan\,x$, is π. The period of $f(x) = tan(2x + 1)$ is different from that of $g(x)$ because the x in $f(x)$ gets multiplied by 2; thus $f(x)$ is moving twice as fast relative to x as is $g(x)$, so its period, the space it takes to go through one cycle, is half as much.

   What about the 1 in $tan(2x + 1)$? That creates a side-to-side shift (more on that later) and is not relevant to the period.

   To understand the change in period, consider some values of the function $h(x) = tan(2x)$, which has the same period as $f(x) = tan(2x + 1)$.

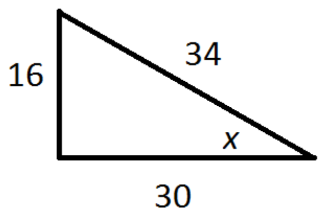
x	$g(x) = tan(x)$	$2x$	$h(x) = tan(2x$
0	0	0	0
$\pi/8$		$\pi/4$	1
$\pi/4$	1	$\pi/2$	undefined
$3\pi/8$		$3\pi/4$	-1
$\pi/2$	undefined	π	0
$5\pi/8$			
$3\pi/4$	-1		
$7\pi/8$			
π	0		

   As you can see, the period of $f(x) = tan(2x)$ is $\pi/2$, half the period of $g(x) = tan\,x$. Adding 1 to $2x$ does not change the period; **the period of** $f(x) = tan(2x + 1)$**is also** $\pi/2$.

2. Find the amplitude of the function $f(x) = 2sin(3x - 1) + \pi$

   The sine and cosine functions move up and down between a minimum and a maximum y −value. For the plain sine function, $g(x) = sin\, x$, those values are -1 and 1, and the function's amplitude is 1, half the difference between the highest value and the lowest. If the whole function gets multiplied by 2, then the amplitude doubles, so the amplitude of $h(x) = 2sin(3x - 1)$ is 2. Adding π to the function will not change the difference between the highest and the lowest y −values, so **the amplitude of** $f(x) = 2sin(3x - 1) + \pi$ **is** 2.

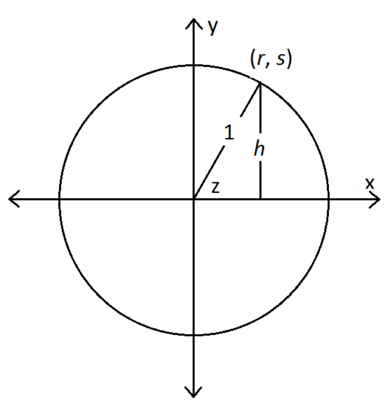
3. Given the right triangle, find $tan\, x$.



   In a right triangle, the tangent of an angle (other than the right angle) is equal to the ratio of the length of the side opposite that angle to the length of the adjacent side.

   In the triangle in the problem, $tan\, x = 16/30 = 8/15$.

4. Given the circle, find $sec\, z$.



   In a *unit circle*, a circle like this one with a radius of 1, the coordinates of the points (r, s) on the circle are $(cos\, z, sin\, z)$.

   Why is that?
   $cos\, z = adjacent/hypotenuse$ is equal to the side of the triangle that lies along the x-axis -- that's also equal to r -- divided by the hypotenuse, which is 1, so $cos\, z = r$. And $sin\, z = opposite/hypotenuse$is equal to $h/1$ and that's also equal to s, the height above the x-axis.

   Now consider that $sec\, z$ is the reciprocal of $cos\, z$. If $cos\, z = r$, then $sec\, z = 1/r$**.**

5. Simplify $cos^2x - sin^2xcos^2x$

   To do this, you need to factor out the common factor and then use the Pythagorean identity $sin^2x + cos^2x = 1$.

$cos^2x - sin^2xcos^2x$	
$cos^2x(1 - sin^2x)$	Factor
$cos^2x(cos^2x)$	Substitute per the Pythagorean identity $sin^2x + cos^2x = 1$; therefore $cos^2x = 1 - sin^2x$
cos^4x	Simplify.

The Pythagorean identities are:

- $1 + tan^2x = sec^2x$
- $cot^2x + 1 = csc^2x$
- $cos^2x + sin^2x = 1$

If you have trouble remembering them, you can reconstruct them. Start with the Pythagorean Theorem, $x^2 + y^2 = r^2$. Then divide each side by x^2, then by y^2, then by r^2 to get the Pythagorean Identities.

$x^2 + y^2 = r^2$	
$\frac{x^2}{x^2} + \frac{y^2}{x^2} = \frac{r^2}{x^2}$	Divide each side by x^2.
$1 + tan^2x = sec^2x$	For each term, substitute either 1 or the term's trig equivalent, as appropriate.
$x^2 + y^2 = r^2$	
$\frac{x^2}{y^2} + \frac{y^2}{y^2} = \frac{r^2}{y^2}$	Divide each side by y^2.
$cot^2x + 1 = csc^2x$	For each term, substitute either 1 or the term's trig equivalent, as appropriate.
$x^2 + y^2 = r^2$	

$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$	Divide each side by r^2.
$cos^2x + sin^2x = 1$	For each term, substitute either 1 or the term's trig equivalent, as appropriate.

6. Which expression is equivalent to $\frac{1-sin^2x}{sinx+1}$?

This is a trig problem, but you can solve it with algebra.

$1 - sin^2x = (1 - sinx)(1 + sinx)$ $=(1 - sinx)(sinx + 1)$	$1 - sin^2x$is a difference of squares. It factors accordingly.
$\frac{1-sin^2x}{sinx+1} = \frac{(1-sinx)(sinx+1)}{sinx+1}$	Substitute the factored form of the numerator.
$\frac{(1-sinx)(sinx+1)}{sinx+1} = 1 - sinx$	Cancel the identical terms.

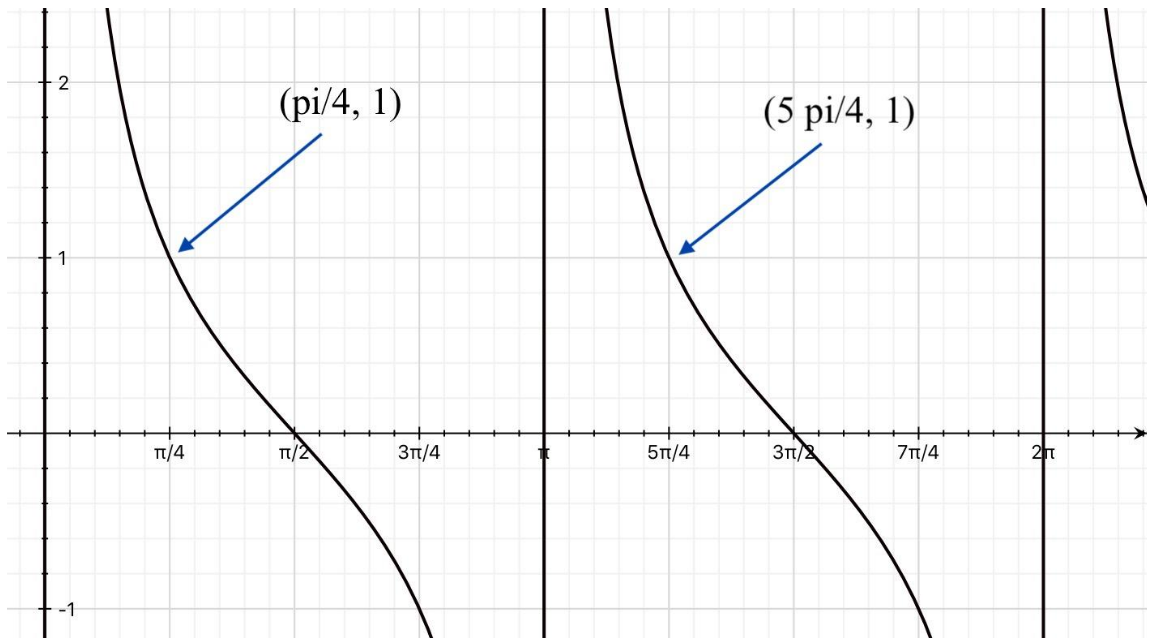
7. Find all solutions of $cot\ z = 1$ on $[0,\ 2\pi)$.

By definition, $cot\ z = \frac{x}{y}$. It equals 1 where $x = y$. That can happen only in quadrants in which x and y have the same sign. Those would be quadrant I, where both x and y are positive, as well as quadrant III, where both x and y are negative.

Where in quadrants I and III does $x = y$? Again, $cot\ z = \frac{x}{y}$ or, in right-triangle trig, $cot\ z = adjacent/opposite$. If $cotz = 1$, that means the two legs of the right triangle are equal in length, so the triangle's base angles are both 45 degrees. So $cot\ z = 1$when $z = 45$degrees, which is $\pi/4$ radians. That's the answer for quadrant I. For the other answer, the one in quadrant III, it helps to remember that the cotangent and tangent curves repeat every π radians. So $cot\ z = 1$ again π radians after the first instance -- that is, at $\pi + \pi/4 = 5\pi/4$.

The answer is $\pi/4$ **,** $5\pi/4$**.**

You may find a sketch helpful. If you know how to sketch the cotangent curve, that could help you either find the answer or check it once you've found it.



Here's a graph of the cotangent function from 0 to 2π. Notice that $y = 1$ at $x = \pi/4$, $x = 5\pi/4$, and nowhere else.

8. Find all solutions of $sin^2 x - 3sin\, x = 4$ on $[0, 2\pi)$.

Start this problem using algebra to find the values of $sin\, x$ involved. Then find the corresponding values of x.

In the first part of this problem, $sin\, x$ is the variable you solve for.

$sin^2 x - 3sin\, x = 4$

$sin^2 x - 3sin\, x - 4 = 0$ Set the problem up to solve it quadratic style.

$(sin\, x - 4)(sin\, x + 1) = 0$ Factor.

$sin\, x - 4 = 0$ $sin\, x + 1 = 0$ Set each factor individually equal to 0.

$sin\, x = 4$ $sin\, x =- 1$ Solve.

Sine cannot equal 4. Reject the extraneous root.

$sin\, x = -\, 1$

Now you need to find all values of x between 0 and 2π that have a sine of -1. That's $3\pi/2$.

9. Sketch the graph of $y = sin(x - \pi/4)$

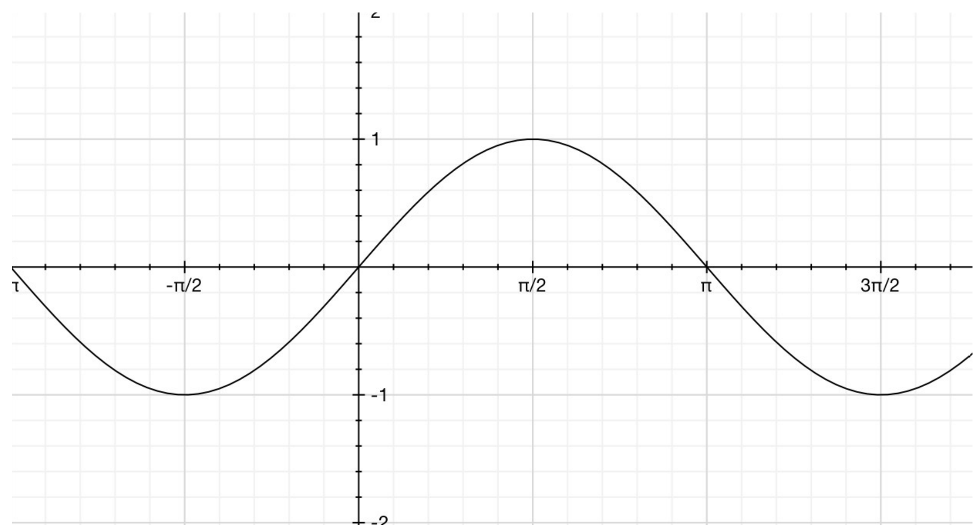
The graph of $y = sin(x - \pi/4)$ is like the graph of $y = sin(x)$ but translated $\pi/4$ units to the right.

If moving to the right because of a subtraction doesn't seem right, let's look at some values:

x	$sin(x)$	$x - \pi/4$	$sin(x - \pi/4)$
$-\pi$	0		
$-3\pi/4$	$-\sqrt{2}/2$	$-\pi$	0
$-\pi/2$	-1	$-3\pi/4$	$-\sqrt{2}/2$
$-\pi/4$	$-\sqrt{2}/2$	$-\pi/2$	-1
0	0	$-\pi/4$	$-\sqrt{2}/2$
$\pi/4$	$\sqrt{2}/2$	0	0
$\pi/2$	1	$\pi/4$	$\sqrt{2}/2$
$3\pi/4$	$\sqrt{2}/2$	$\pi/2$	1
π	0	$3\pi/4$	$\sqrt{2}/2$
$5\pi/4$	$-\sqrt{2}/2$	π	0
$3\pi/2$	-1	$5\pi/4$	$-\sqrt{2}/2$
$7\pi/4$	$-\sqrt{2}/2$	$3\pi/2$	-1
2π	0	$7\pi/4$	$-\sqrt{2}/2$
		2π	0

Note that at every value of x, $x - \pi/4$ takes the value that x took a distance of $\pi/4$ to the left. That has the effect of pushing the curve forward $\pi/4$ units.

How does it look on a graph?



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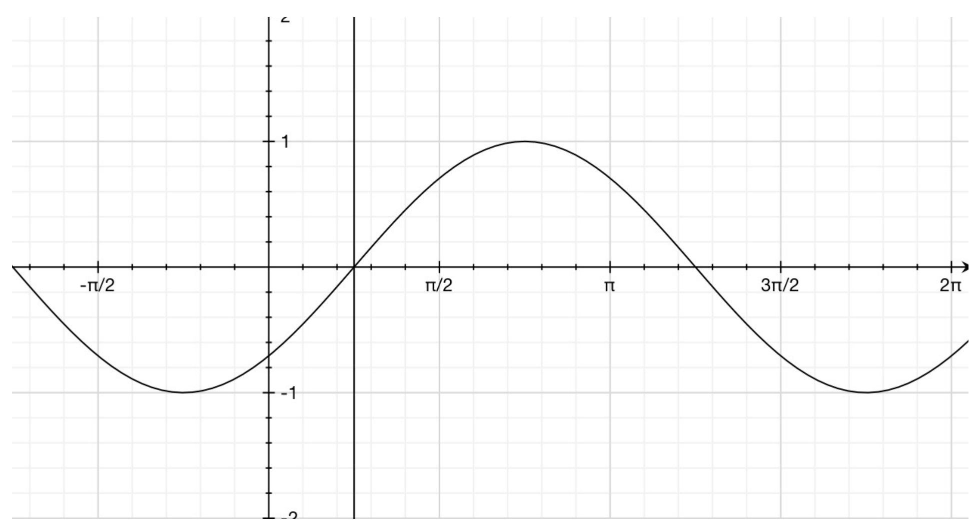
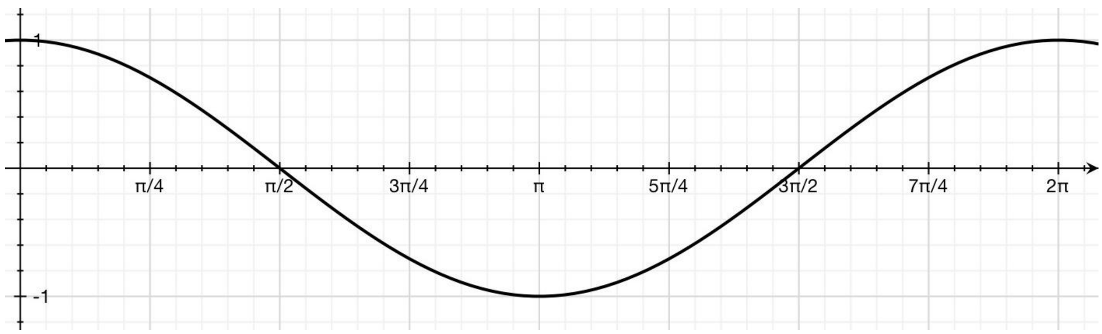


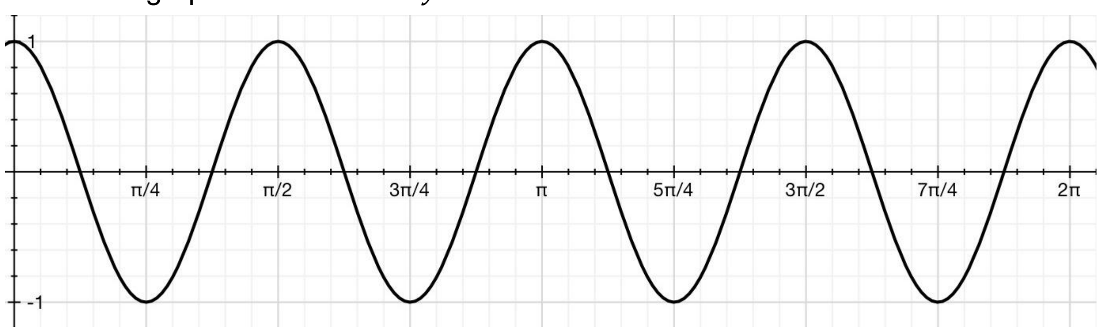
On top is the graph of $y = sin\, x$ and below that is the graph of $y = sin\,(x - \pi/4)$. In the bottom graph is also the vertical line $x = \pi/4$. Note that the line $x = \pi/4$ is in the same position with respect to the graph of $y = sin\,(x - \pi/4)$ as is the y-axis with respect to the graph of $y = sin\, x$ and that the y-axis in the graph in the graph $y = sin\,(x - \pi/4)$ is a distance $\pi/4$ to the left if where it is in the graph $y = sin\, x$. In other words, when the curve changes from $y = sin\, x$ to $y = sin\,(x - \pi/4)$, you can think of the y-axis moving $\pi/4$ to the left while the curve stands still. Subtracting $\pi/4$ from x moves the frame of reference $\pi/4$ to the left, which has the same effect as moving the curve $\pi/4$ to the right with respect to the frame of reference.

10. Sketch the graph of $y = 2cos(4x)$.

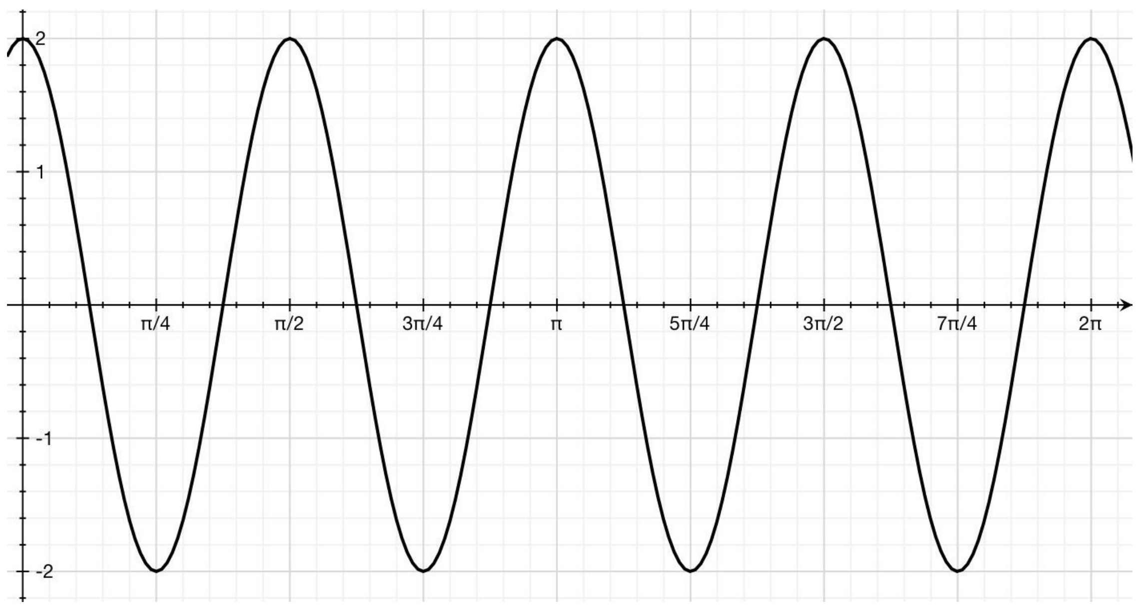
Start with the graph of $y = cosx$ and make changes to it to get to $y = 2cos(4x)$.



Above is a graph of the function $y = cosx$.



Above is a graph of the function $y = cos(4x)$. As you can see, multiplying x by 4 has horizontally compressed the graph of $y = cosx$. The compressed graph goes through four cycles between 0 and 2π while the original graph goes through just one cycle in that space. Horizontal compression is explained in the discussion of problem 1.

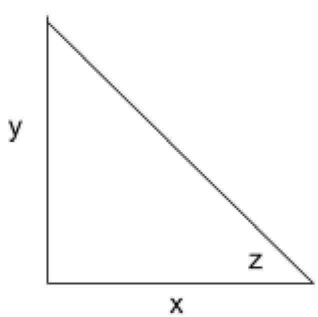


Above is a graph of the function $y = 2cos4x$. As you can see, the graph of $y = cos4x$ has been vertically doubled.

11. Given that $sin(A + B) = sinAcosB + cosAsinB$, find $sin(2x)$

    In $sin(2x)$, x plays the roles that both A and B play in $sin(A + B)$. So substitute x for both A and B in the formula:
    $sin(A + B) = sinAcosB + cosAsinB$
    $sin(2x) = sinxcosx + cosxsinx$
    $sin(2x) = 2sinxcosx$

12. A wire runs from the top of a pole that is y feet tall to the ground. The wire touches the ground a distance x feet from the pole's base. The wire makes an angle z with the top of the pole. What is y in terms of x and z?

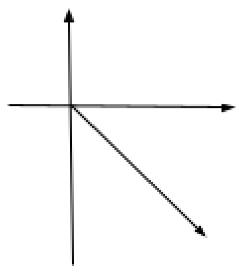


Make a sketch.

From the sketch you can see that $tan\, z = y/x$, or $y = xtanz$.

13. Evaluate $sin\left(tan^{-1}(-\ 1)\right)$.

You are asked to find the sin of the arctan of -1. That is, you need to find the sine of the angle that has a tangent of -1 -- not just any angle with a tangent of -1, but the one within the range of the arctan function, which is the interval $(-\ \pi/2,\ \pi/2)$.



That angle is $-\ \pi/4$. What's its sine? The reference angle for $-\ \pi/4$ is $\pi/4$, so the absolute value of $sin(-\ \pi/4)$ is the same as that of $sin\pi/4$: $\sqrt{2}/2$. What about the sign? You are working in quadrant III, where sine is negative. So the answer is $-\sqrt{2}/2$.

For problems like this, involving a trig function of an inverse trig function, or an inverse trig function of a trig function, the tricky issue is range. Each of the inverse trig functions arcsin, arccos, and arctan (aka sin^{-1}, cos^{-1}, and tan^{-1}) has a range that covers just part of the coordinate plane.

Function	$sin^{-1}x$	$cos^{-1}x$	$tan^{-1}x$
Range	$[-\ \pi/2,\ \pi/2]$	$[0,\ \pi]$	$(-\ \pi/2,\ \pi/2)$

The ranges are set up so that every y-value in the original function (sine, cosine, or tangent) is represented by one and only one x-value in the inverse function (arcsin, arccos, or arctan). Thus every possible y −value of, say, $sin\,x$ has a corresponding x −value in the arcsin function. The same is true for the cosine and tangent functions. At the same time, for the original functions every value of y in the original function has no more than one corresponding x −value; thus, for the inverse relations, every value of x has only one corresponding value of y and therefore the inverse relations are functions.

Once you understand that the ranges of arcsine and arccosine are chosen so that those relations are functions, you may find the ranges easier to remember (or to figure out on the fly). For tangent it's a little tricker, because there are more possible ranges that could work. But the range $(-\ \pi/2,\ \pi/2)$ has an added advantage: It avoids the asymptotes, which are sticky places worth avoiding.

You may get a problem like this:

Evaluate $sin^{-1}(sin(\pi))$.

If you just say these functions undo each other so the answer must be π, you will have the wrong answer, because π is outside the range of the arcsin function. You have to go through the steps, from the inside out: The $sin\pi = 0$ and $sin^{-1}(0) = 0$, so $sin^{-1}(sin(\pi)) = 0$.

14. Evaluate $csc^{-1}(2)$.

The range of the inverse cosecant function is $\left[- \pi/2,\ 0) \cup (0,\ \pi/2\right]$. So the question amounts to "What angle between $-\ \pi/2$ and $\pi/2$ not equal to 0 has a cosecant of 2?" That's the angle whose sine is the reciprocal of 2, ½: $\pi/6$.

15. Identify the domain and range of $f(x) = 3sinx$.

The domain of both the sine function and the cosine function is all real numbers, represented as $(-\ \infty,\ \infty)$. That is, you can find the sine or cosine of any real number. So the domain of sine of $f(x) = 3sinx$ is all real numbers, or $(-\ \infty,\ \infty)$.

The range of both the sine function and the cosine function is $[-\ 1,\ 1]$. The function you are asked about, $f(x) = 3sinx$, is 3 times the sine function; its range is three times the range of the sine function: $[-\ 3,\ 3]$.

16. What is the equation of an asymptote of $g(\Theta) = cot\Theta$?

The tangent and cotangent functions have asymptotes where their denominators equal 0. Cotangent is defined as x/y; its denominator is 0 at $y = 0$; this happens at $\Theta = 0,\ \pi$, and multiples of π. So the correct answer is $\Theta = 0$, $\Theta = \pi$, or $\Theta =$ any multiple of π.