3 Determinants

3.1 Introduction to Determinants

It was remarked in a previous section that if the determinant $ad - bc$ of the $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonzero, then $A$ is invertible (and vice-versa). To see why this is true, we assume $a \neq 0$ so that $A$ has a pivot in the first row, then perform some row operations on $A$:

$$
\begin{align*}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} & \rightarrow ar_2 \\
\begin{bmatrix} a & b \\ ac & ad \end{bmatrix} & \rightarrow r_2 \rightarrow r_2 - cr_1 \\
\begin{bmatrix} a & b \\ c & d - bc \end{bmatrix} & \rightarrow \\
\end{align*}
$$

In order for this matrix to be invertible, we need a pivot in the second row, which happens if and only if $ad - bc \neq 0$. Notice that this quantity $ad - bc$ is a number that is formed using every entry of the matrix, and it tells us very useful information about the matrix. This same idea generalizes to larger matrices; i.e. associated to every $n \times n$ matrix $A$, there is a determinant $\det A$ (sometimes denoted $\Delta$) involving every entry of $A$ that tells us whether this matrix is invertible.

**Notation** For any square $n \times n$ matrix $A$, we let $A_{ij}$ be the $(n - 1) \times (n - 1)$ matrix obtained by deleting the $i$th row and the $j$th column.

**Example** Let $A$ be the $3 \times 3$ matrix

$$
A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.
$$

Then

$$
A_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}, \quad \text{and} \quad A_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}.
$$

**Definition** For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of $n$ terms of the form $\pm a_{1j} \det A_{1j}$, with the plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of $A$. In symbols,

$$
\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n}.
$$

In summation notation, we write

$$
\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}.
$$

**Remark** This definition is recursive. For example, computing the determinant of a $3 \times 3$ matrix is dependent on knowing how to compute the determinant of a $2 \times 2$ matrix. More generally, computing the determinant of an $n \times n$ matrix is dependent on knowing how to compute an $(n - 1) \times (n - 1)$ determinant, which is dependent on knowing how to compute an $(n - 2) \times (n - 2)$ determinant, etc. In practice, we usually let a computer handle such a computation for large matrices.

**Example** We compute the determinant of

$$
A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.
$$
Using the definition, we have

\[ \det A = 1 \cdot \det A_{11} - 2 \cdot \det A_{12} + 3 \cdot \det A_{13} \]

\[ = \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2 \det \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3 \det \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} \]

\[ = (5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) \]

\[ = (45 - 48) - 2(36 - 42) + 3(32 - 35) \]

\[ = -3 - 2(-6) + 3(-3) \]

\[ = -3 + 12 - 9 = 0. \]

This tells us that \( A \) is not invertible (though we won't provide an explanation until the next section).

In the definition of the determinant given above, it may seem as though there is something special about expanding along the first row. This turns out to be false. That is, we can compute the determinant by a process called cofactor expansion across any row or any column. The power of this is that it allows for fast computation of the determinant when a particular row or column has lots of zeros.

**Notation** We often use the notation \(|\cdot|\) instead of \(\det\) to indicate that a determinant is being computed. For example

\[ \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}. \]

**Definition** Given an \( n \times n \) matrix \( A = [a_{ij}] \), the \((i, j)\)-cofactor of \( A \) is the number \( C_{ij} \) given by

\[ C_{ij} = (-1)^{i+j} \det A_{ij}. \]

With this definition, we see that

\[ \det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}. \]

This is called the cofactor expansion along the first row of \( A \).

**Theorem** The determinant of an \( n \times n \) matrix \( A \) can be computed by a cofactor expansion across any row or down any column. The expansion across the \( i \)th row using the cofactors is

\[ \det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}. \]

Similarly, the expansion down the \( j \)th column using cofactors is

\[ \det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}. \]

**Note** The plus or minus sign in the \((i, j)\)-cofactor depends on the position of \( a_{ij} \) in the matrix. The factor \((-1)^{i+j}\) determines a “checkerboard” pattern as follows:

\[ \begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \]

**Example** We use cofactor expansion across the third row of the following matrix to simplify the computation of the determinant:

\[ A = \begin{bmatrix} 2 & 1 & 0 \\ 5 & 3 & 1 \\ 0 & 2 & 0 \end{bmatrix}. \]
We see that
\[
\det A = 0 \cdot \det A_{31} - 2 \cdot \det A_{32} + 0 \cdot \det A_{33}
\]
\[
= -2 \begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix}
\]
\[
= -2(2 \cdot 1 - 0 \cdot 5)
\]
\[
= -4.
\]

We conclude that this matrix is invertible. Likewise, we could have expanded along the third column, as follows:
\[
\det A = 0 \cdot \det A_{13} - 1 \cdot \det A_{23} + 0 \cdot \det A_{33}
\]
\[
= -1 \cdot \det A_{23}
\]
\[
= - \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}
\]
\[
= -(2 \cdot 2 - 1 \cdot 0)
\]
\[
= -4.
\]

Notice that we get the same answer.

**Theorem** If \( A \) is a triangular matrix, then \( \det A \) is the product of the entries on the main diagonal of \( A \).

**Example** We compute the determinant of
\[
A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}.
\]

By the theorem,
\[
\det A = 1 \cdot 6 \cdot 1 \cdot 4 \cdot 6 = 144.
\]

**Note** In applications, matrices of size, say, \( 25 \times 25 \) are relatively small. Yet computing the determinant of such a matrix using the method outlined in this section would take approximately \( 25! = 25 \cdot 24 \cdot 23 \cdot \cdots \cdot 1 \) multiplications, which is roughly \( 1.5 \times 10^{25} \). This would take a computer many years to perform (the textbook gives an estimate of over 500,000 years, assuming the computer performs one trillion multiplications per second). There are other ways to significantly shorten the computation of a determinant.

**Example** The following example is to be worked on in class (time-permitting):

Compute
\[
\begin{vmatrix} 1 & -4 & 2 & -1 \\ 0 & 4 & 0 & 2 \\ 3 & -2 & 0 & 5 \\ 0 & 1 & 0 & 7 \end{vmatrix}.
\]